

MAXIMAL SOBOLEV REGULARITY FOR SOLUTIONS OF ELLIPTIC EQUATIONS IN INFINITE DIMENSIONAL BANACH SPACES ENDOWED WITH A WEIGHED GAUSSIAN MEASURE

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ABSTRACT. Let X be a separable Banach space endowed with a non-degenerate centered Gaussian measure μ . The associated Cameron–Martin space is denoted by H . Let $\nu = e^{-U}\mu$, where $U : X \rightarrow \mathbb{R}$ is a sufficiently regular convex and continuous function. In this paper we are interested in the $W^{2,2}$ regularity of the weak solutions of elliptic equations of the type

$$\lambda u - L_\nu u = f,$$

where $\lambda > 0$, $f \in L^2(X, \nu)$ and L_ν is the self-adjoint operator associated with the quadratic form

$$(\psi, \varphi) \mapsto \int_X \langle \nabla_H \psi, \nabla_H \varphi \rangle_H d\nu \quad \psi, \varphi \in W^{1,2}(X, \nu).$$

1. INTRODUCTION

Let X be a separable Banach space with norm $\|\cdot\|_X$, endowed with a non-degenerate centered Gaussian measure μ . The associated Cameron–Martin space is denoted by H , its inner product by $\langle \cdot, \cdot \rangle_H$ and its norm by $|\cdot|_H$. The spaces $W^{1,p}(X, \mu)$ and $W^{2,p}(X, \mu)$ for $p \geq 1$ are the classical Sobolev spaces of the Malliavin calculus (see [8, Chapter 5]).

The aim of this paper is to study the solutions of the equation

$$(1.1) \quad \lambda u - L_\nu u = f$$

where $\lambda > 0$, ν is a measure of the form $e^{-U}\mu$ with $U : X \rightarrow \mathbb{R}$ a convex and continuous function, $f \in L^2(X, \nu)$ and L_ν is the operator associated to the quadratic form

$$(\psi, \varphi) \mapsto \int_X \langle \nabla_H \psi, \nabla_H \varphi \rangle_H d\nu \quad \psi, \varphi \in W^{1,2}(X, \nu),$$

where $\nabla_H \psi$ represent the gradient along H of ψ and $W^{1,2}(\Omega, \nu)$ is the Sobolev space on Ω associate to the measure ν (see Section 2).

We need to clarify what we mean with *solution* of problem (1.1). We say that $u \in W^{1,2}(X, \nu)$ is a *weak solution* of equation (1.1) if

$$\lambda \int_X u \varphi d\nu + \int_X \langle \nabla_H u, \nabla_H \varphi \rangle_H d\nu = \int_X f \varphi d\nu \quad \text{for every } \varphi \in \mathcal{FC}_b^\infty(X).$$

Notice that the weak solution is just $R(\lambda, L_\nu)f$, the resolvent of L_ν .

In the finite dimensional case, existence, uniqueness and maximal regularity of the solution of equation (1.1) have been widely studied. Indeed in the case of the standard Gaussian measure in \mathbb{R}^n , the operator L_ν reads as

$$L_\nu u(\xi) = \Delta u(\xi) - \langle \text{grad } U(\xi) + \xi, \text{grad } u(\xi) \rangle \quad u \in \mathcal{C}_b^2(\mathbb{R}^n),$$

so that, if U is smooth, L_ν is an elliptic operator with smooth, although possibly unbounded, coefficients. See for example [10], [18] and [21]. In the infinite dimensional case maximal $W^{2,2}$ regularity results are known

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when X is a separable Hilbert space. See for example [11], where U is assumed to be bounded from below. In the general Banach spaces case some results are known about equation (1.1), but we do not know any $W^{2,2}$ regularity result. See for example [1], where a much larger class of operator is studied.

In order to state the results of this paper we need some hypotheses on the weighted measure ν .

Hypothesis 1.1. $U : X \rightarrow \mathbb{R}$ is a convex and continuous function belonging to $W^{1,t}(X, \mu)$ for some $t > 3$. We set $\nu := e^{-U} \mu$.

The assumption $t > 3$ may sound strange, but it is needed to define the weighted Sobolev spaces $W^{1,2}(X, \nu)$. Indeed observe that if U satisfies Hypothesis 1.1, then it satisfies [15, Hypothesis 1.1] since, by [2, Lemma 7.5], e^{-U} belongs to $W^{1,r}(X, \mu)$ for every $r < t$. Then following [15] it is possible to define the space $W^{1,2}(X, \nu)$ as the domain of the closure of the gradient operator along H .

The main result of this paper is the following theorem.

Theorem 1.2. *Let U be a function satisfying Hypothesis 1.1, let $\lambda > 0$ and $f \in L^2(X, \nu)$. Then equation (1.1) has a unique weak solution $u \in W^{2,2}(X, \nu)$. Moreover u satisfies*

$$(1.2) \quad \|u\|_{L^2(X, \nu)} \leq \frac{1}{\lambda} \|f\|_{L^2(X, \nu)}; \quad \|\nabla_H u\|_{L^2(X, \nu; H)} \leq \frac{1}{\sqrt{\lambda}} \|f\|_{L^2(X, \nu)};$$

$$(1.3) \quad \|\nabla_H^2 u\|_{L^2(X, \nu; \mathcal{H}_2)} \leq \sqrt{2} \|f\|_{L^2(X, \nu)}.$$

where ∇_H^2 is defined in Section 2 and \mathcal{H}_2 is the space of the Hilbert–Schmidt operators in H .

The paper is organized in the following way: in section 2 we recall some basic definitions and we fix the notations. Section 3 is dedicated to modify a standard tool in the theory of convex functions on Hilbert spaces: the Moreau–Yosida approximations (see [4] and [9]). In Section 4 we recall known results about finite dimensional elliptic and parabolic equations that we will use. In Section 5 we study the case in which $\nabla_H U$ is a H -Lipschitz function. Then we prove that equation (1.1) admits a strong solution in the following sense:

Definition 1.3. A function $u \in L^2(X, \nu)$ is a *strong solution of equation (1.1)* if there exists a sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq \mathcal{FC}_b^3(X)$ such that u_n converges to u in $L^2(X, \nu)$ and

$$L^2(X, \nu)\text{-}\lim_{n \rightarrow +\infty} \lambda u_n - L_\nu u_n = f.$$

Moreover a sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq \mathcal{FC}_b^3(X)$ satisfying the above conditions is called a *strong solution sequence* for u .

We conclude the section proving Theorem 1.2. In Section 6 we will recall some results about the divergence operator on weighted Gaussian spaces and we will show that if U satisfies Hypothesis 1.1 and $\nabla_H U$ is H -Lipschitz, then $D(L_\nu) = W^{2,2}(X, \nu)$ and

$$\|u\|_{D(L_\nu)} \leq \|u\|_{W^{2,2}(X, \nu)} \leq (2 + \sqrt{2}) \|u\|_{D(L_\nu)},$$

where $\|\cdot\|_{D(L_\nu)}$ is the graph norm in $D(L_\nu)$, i.e. for $u \in D(L_\nu)$

$$(1.4) \quad \|u\|_{D(L_\nu)} := \|u\|_{L^2(X, \nu)} + \|L_\nu u\|_{L^2(X, \nu)}.$$

See Section 2 for the definition of $W^{2,2}(X, \nu)$. In the final section we show how our results can be applied to some examples. In our examples X will be $\mathcal{C}_0[0, 1] = \{f \in \mathcal{C}[0, 1] \mid f(0) = 0\}$, endowed with the classical Wiener measure P^W . First we consider, for $f \in \mathcal{C}_0[0, 1]$,

$$U(f) = \int_0^1 f^2(\xi) d\xi$$

and we show that U is a weight bounded from below, satisfies Hypothesis 1.1 and $\nabla_H U$ is H -Lipschitz. In Example 7.2 we consider the following function, for $f \in \mathcal{C}_0[0, 1]$,

$$U(f) = F(f) + f(1)$$

where $F(f) = \max_{\xi \in [0, 1]} f(\xi)$. We show that U satisfies Hypothesis 1.1 although it is unbounded, both from above and from below.

2. NOTATIONS AND PRELIMINARIES

We will denote by X^* the topological dual of X . We recall that $X^* \subseteq L^2(X, \mu)$. The linear operator $R_\mu : X^* \rightarrow X^{**}$ defined by the formula

$$R_\mu x^*(y^*) = \int_X x^*(x) y^*(x) d\mu(x)$$

is called the covariance operator of μ . We denote by X_μ^* the closure of X^* in $L^2(X, \mu)$. The covariance operator R_μ can be extended by continuity to the space X_μ^* . By [8, Lemma 2.4.1] for every $h \in H$ there exists a unique $g \in X_\mu^*$ with $h = R_\mu(g)$, in this case we set

$$(2.1) \quad \widehat{h} := g.$$

Throughout the paper we fix an orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$ of H such that \widehat{e}_i belongs to X^* , for every $i \in \mathbb{N}$. Such basis exists by [8, Corollary 3.2.8(ii)].

We say that a function $f : X \rightarrow \mathbb{R}$ is *differentiable along H at x* if there is $v \in H$ such that

$$\lim_{t \rightarrow 0} \frac{f(x + th) - f(x)}{t} = \langle v, h \rangle_H \quad \text{uniformly for } h \in H \text{ with } |h|_H = 1.$$

In this case the vector $v \in H$ is unique and we set $\nabla_H f(x) := v$, moreover for every $k \in \mathbb{N}$ the derivative of f in the direction of e_k exists and it is given by

$$(2.2) \quad \partial_k f(x) := \lim_{t \rightarrow 0} \frac{f(x + te_k) - f(x)}{t} = \langle \nabla_H f(x), e_k \rangle_H.$$

We denote by \mathcal{H}_2 the space of the Hilbert–Schmidt operators in H , that is the space of the bounded linear operators $A : H \rightarrow H$ such that $\|A\|_{\mathcal{H}_2}^2 = \sum_i |Ae_i|_H^2$ is finite (see [12]). We say that a function $f : X \rightarrow \mathbb{R}$ is *two times differentiable along H at x* if it is differentiable along H at x and $A \in \mathcal{H}_2$ exists such that

$$H\text{-}\lim_{t \rightarrow 0} \frac{\nabla_H f(x + th) - \nabla_H f(x)}{t} = Ah \quad \text{uniformly for } h \in H \text{ with } |h|_H = 1.$$

In this case the operator A is unique and we set $\nabla_H^2 f(x) := A$. Moreover for every $i, j \in \mathbb{N}$ we set

$$(2.3) \quad \partial_{ij} f(x) := \lim_{t \rightarrow 0} \frac{\partial_j f(x + te_i) - \partial_j f(x)}{t} = \langle \nabla_H^2 f(x) e_j, e_i \rangle_H.$$

For $k \in \mathbb{N} \cup \{\infty\}$, we denote by $\mathcal{FC}_b^k(X)$ the space of the cylindrical function of the type $f(x) = \varphi(x_1^*(x), \dots, x_n^*(x))$ where $\varphi \in \mathcal{C}_b^k(\mathbb{R}^n)$ and $x_1^*, \dots, x_n^* \in X^*$ and $n \in \mathbb{N}$. We remark that $\mathcal{FC}_b^\infty(X)$ is dense in $L^p(X, \nu)$ for all $p \geq 1$ (see [15, Proposition 3.6]). We recall that if $f \in \mathcal{FC}_b^2(X)$, then $\partial_{ij} f(x) = \partial_{ji} f(x)$ for every $i, j \in \mathbb{N}$ and $x \in X$.

The Gaussian Sobolev spaces $W^{1,p}(X, \mu)$ and $W^{2,p}(X, \mu)$, with $p \geq 1$, are the completions of the *smooth cylindrical functions* $\mathcal{FC}_b^\infty(X)$ in the norms

$$\begin{aligned} \|f\|_{W^{1,p}(X, \mu)} &:= \|f\|_{L^p(X, \mu)} + \left(\int_X |\nabla_H f(x)|_H^p d\mu(x) \right)^{\frac{1}{p}}; \\ \|f\|_{W^{2,p}(X, \mu)} &:= \|f\|_{W^{1,p}(X, \mu)} + \left(\int_X \|\nabla_H^2 f(x)\|_{\mathcal{H}_2}^p d\mu(x) \right)^{\frac{1}{p}}. \end{aligned}$$

Such spaces can be identified with subspaces of $L^p(X, \mu)$ and the (generalized) gradient and Hessian along H , $\nabla_H f$ and $\nabla_H^2 f$, are well defined and belong to $L^p(X, \mu; H)$ and $L^p(X, \mu; \mathcal{H}_2)$, respectively. For more informations see [8, Section 5.2].

Now we consider $\nabla_H : \mathcal{FC}_b^\infty(X) \rightarrow L^p(X, \nu; H)$. This operator is closable in the norm of $L^p(X, \nu)$ whenever $p > \frac{t-1}{t-2}$ and Hypothesis 1.1 holds (see [15, Definition 4.3]). For such p we denote by $W^{1,p}(X, \nu)$ the domain of its closure in $L^p(X, \nu)$.

Assume Hypotesis 1.1 holds. We shall use the integration by parts formula (see [15, Lemma 4.1]) for $\varphi \in W^{1,p}(X, \mu)$ with $p > \frac{t-1}{t-2}$:

$$(2.4) \quad \int_X \partial_k \varphi d\nu = \int_X \varphi (\partial_k U + \widehat{e}_k) d\nu \quad \text{for every } k \in \mathbb{N},$$

where \widehat{e}_k is defined in formula (2.1).

In order to define the spaces $W^{2,p}(X, \nu)$, we need to prove the closability of the operator (∇_H, ∇_H^2) in $L^p(X, \nu)$.

Proposition 2.1. *Assume Hypotesis 1.1 holds. For every $p \geq \frac{t-1}{t-2}$, the operator*

$$(\nabla_H, \nabla_H^2) : \mathcal{FC}_b^\infty(X) \rightarrow L^p(X, \nu; H) \times L^p(X, \nu; \mathcal{H}_2)$$

is closable in $L^p(X, \nu)$. The closure will be still denoted by (∇_H, ∇_H^2) .

Proof. Let $\{\varphi_k\}_{k \in \mathbb{N}} \subseteq \mathcal{FC}_b^\infty(X)$ be such that $\varphi_k \rightarrow 0$ in $L^p(X, \nu)$, $\nabla_H \varphi_k \rightarrow F$ in $L^p(X, \nu; H)$, and $\nabla_H^2 \varphi_k \rightarrow \Phi$ in $L^p(X, \nu; \mathcal{H}_2)$ as $k \rightarrow +\infty$. By [15, Proposition 4.2] $F = 0$ ν -a.e. Let $\psi \in \mathcal{FC}_b^\infty(X)$, then by the integration by parts formula (formula (2.4)) we get

$$\int_X \psi \partial_{ij} \varphi_k d\nu = \int_X \psi \partial_j \varphi_k (\partial_i U + \widehat{e}_i) d\nu - \int_X \partial_j \psi \partial_i \varphi_k d\nu.$$

We remark that

$$\lim_{k \rightarrow +\infty} \int_X \psi \partial_{ij} \varphi_k d\nu = \int_X \psi \langle \Phi e_i, e_j \rangle_H d\nu.$$

Moreover

$$\lim_{k \rightarrow +\infty} \int_X \partial_j \psi \partial_i \varphi_k d\nu = 0 \quad \text{and} \quad \lim_{k \rightarrow +\infty} \int_X \widehat{e}_i \partial_j \varphi_k \psi d\nu = 0.$$

Then $\int_X \psi \langle \Phi e_i, e_j \rangle_H d\nu = 0$ for every $\psi \in \mathcal{FC}_b^\infty(X)$. So $\Phi = 0$ μ -a.e. since $\mathcal{FC}_b^\infty(X)$ is dense in $L^{p'}(X, \nu)$ (see [15, Proposition 3.6]). \square

We are now able to define the Sobolev spaces $W^{2,p}(X, \nu)$.

Definition 2.2. Assume Hypotesis 1.1 holds. For $p > \frac{t-1}{t-2}$ we denote by $W^{2,p}(X, \nu)$ the domain of closure of the operator $(\nabla_H, \nabla_H^2) : \mathcal{FC}_b^\infty(X) \rightarrow L^p(X, \nu; H) \times L^p(X, \nu; \mathcal{H}_2)$ in $L^p(X, \nu)$.

We remark that if $t > 3$, i.e. when Hypotesis 1.1 holds, then $2 > \frac{t-1}{t-2}$.

We remind the reader that, if U satisfies Hypotesis 1.1 and belongs $W^{2,t}(X, \mu)$, where t is the same as in Hypotesis 1.1, then by the integration by parts formula (formula (2.4)), and [15, Proposition 5.3] (see also Proposition 6.1) we get for every $u \in \mathcal{FC}_b^2(X)$

$$(2.5) \quad L_\nu u = \sum_{i=1}^{+\infty} \partial_{ii} u - \sum_{i=1}^{+\infty} (\partial_i U + \widehat{e}_i) \partial_i u,$$

where the series converges in $L^2(X, \nu)$.

Finally we recall the following corollary of the Hahn–Banach theorem (see [2, Lemma 7.5]).

Proposition 2.3. *Let $g : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex and lower semicontinuous function and let $r \in \mathbb{R}$ such that there is $x_0 \in X$ with $g(x_0) > r$. Then there exists $x^* \in X^*$ such that for every $x \in X$*

$$g(x) \geq x^*(x - x_0) + r.$$

3. MOREAU–YOSIDA APPROXIMATIONS ALONG H

In this section we will modify a classical tool in the theory of convex functions in Hilbert spaces: the Moreau–Yosida approximations. For a classical treatment of the Moreau–Yosida approximations in Hilbert spaces we refer to [4, Section 12.4].

Throughout this section $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is a convex and $\|\cdot\|_X$ -lower semicontinuous function. Let $\alpha > 0$ and define the *Moreau–Yosida approximation along H* as

$$(3.1) \quad f_\alpha(x) = \inf \left\{ f(x+h) + \frac{1}{2\alpha} |h|_H^2 \mid h \in H \right\}.$$

Proposition 3.1. *Let $x \in X$ and $\alpha > 0$. The function $g_{\alpha,x} : H \rightarrow \mathbb{R}$ defined as*

$$g_{\alpha,x}(h) = f(x+h) + \frac{1}{2\alpha} |h|_H^2,$$

is convex, $|\cdot|_H$ -lower semicontinuous and it has a unique global minimum point $P(x, \alpha) \in H$. Moreover $g_{\alpha,x}$ is coercive, i.e.

$$\lim_{|h|_H \rightarrow +\infty} g_{\alpha,x}(h) = +\infty.$$

Proof. Convexity is trivial. Let $H\text{-}\lim_{n \rightarrow +\infty} h_n = h$. Since H is continuously embedded in X , $X\text{-}\lim_{n \rightarrow +\infty} h_n = h$. By the fact that f is $\|\cdot\|_X$ -lower semicontinuous, we get

$$f(x+h) \leq \liminf_{n \rightarrow +\infty} f(x+h_n).$$

So $g_{\alpha,x}$ is $|\cdot|_H$ -lower semicontinuous. By Proposition 2.3, for every $x \in X$ there exist $h(x) \in H$ and $\eta \in \mathbb{R}$ such that $f(x+h) \geq \langle h, h(x) \rangle_H + \eta$ for every $h \in H$. So we get

$$\begin{aligned} \lim_{|h|_H \rightarrow +\infty} g_{\alpha,x}(h) &= \lim_{|h|_H \rightarrow +\infty} \left(f(x+h) + \frac{1}{2\alpha} |h|_H^2 \right) \geq \lim_{|h|_H \rightarrow +\infty} \left(\langle h, h(x) \rangle_H + \eta + \frac{1}{2\alpha} |h|_H^2 \right) \geq \\ &\geq \lim_{|h|_H \rightarrow +\infty} \left(-|h|_H |h(x)|_H + \eta + \frac{1}{2\alpha} |h|_H^2 \right) = +\infty. \end{aligned}$$

Since $g_{\alpha,x}$ is convex, $|\cdot|_H$ -lower semicontinuous and coercive, the set

$$A_{\alpha,x} := \{p \in H \mid g_{\alpha,x}(p) = \inf \{g_{\alpha,x}(h) \mid h \in H\}\}$$

is nonempty (see [4, Proposition 11.14]). We claim that $A_{\alpha,x}$ is a singleton. Indeed, by contradiction, assume that $p_1, p_2 \in A_{\alpha,x}$ are such that $p_1 \neq p_2$. Using the strict convexity of $|\cdot|_H$

$$\begin{aligned} g_{\alpha,x} \left(\frac{p_1 + p_2}{2} \right) &= f \left(x + \frac{p_1 + p_2}{2} \right) + \frac{1}{2\alpha} \left| \frac{p_1 + p_2}{2} \right|_H^2 < \\ &< \frac{1}{2} \left(f(x+p_1) + \frac{1}{2\alpha} |p_1|_H^2 \right) + \frac{1}{2} \left(f(x+p_2) + \frac{1}{2\alpha} |p_2|_H^2 \right) \leq \frac{1}{2} g_{\alpha,x}(p_1) + \frac{1}{2} g_{\alpha,x}(p_2) = \\ &= \inf \{g_{\alpha,x}(h) \mid h \in H\}, \end{aligned}$$

a contradiction. \square

Proposition 3.2. *For every $x \in X$ we have $f_\alpha(x) \nearrow f(x)$ as $\alpha \rightarrow 0^+$. In particular $f_\alpha(x) \leq f(x)$ for every $\alpha > 0$ and $x \in X$.*

Proof. Monotonicity of f_α is obvious. Let $S(x) := \lim_{\alpha \rightarrow 0^+} f_\alpha(x) = \sup_{\alpha \in (0,1)} f_\alpha(x)$. Since $f_\alpha(x) \leq f(x)$ we have $S(x) \leq f(x)$. If $S(x) = +\infty$ then there is nothing to prove.

Assume $S(x) < +\infty$. We just need to prove that $S(x) \geq f(x)$. By monotonicity we get

$$\{P(x, \alpha) \mid \alpha \in (0, 1)\} \subseteq \{h \in H \mid g_{1,x}(h) \leq S(x)\}.$$

By Proposition 3.1 the set $\{P(x, \alpha) \mid \alpha \in (0, 1)\} \subseteq H$ is bounded. Let

$$c(x) = \sup \{|P(x, \alpha)|_H \mid \alpha \in (0, 1)\}.$$

By Proposition 2.3, for every $x \in X$ there exist $h(x) \in H$ and $\eta \in \mathbb{R}$ such that $f(x+h) \geq \langle h, h(x) \rangle_H + \eta$ for every $h \in H$. Then, for every $\alpha \in (0, 1)$, we have

$$\begin{aligned} S(x) &\geq f_\alpha(x) = f(x + P(x, \alpha)) + \frac{1}{2\alpha} |P(x, \alpha)|_H^2 \geq \langle P(x, \alpha), h(x) \rangle_H + \eta + \frac{1}{2\alpha} |P(x, \alpha)|_H^2 \geq \\ &\geq -|P(x, \alpha)|_H |h(x)|_H + \eta + \frac{1}{2\alpha} |P(x, \alpha)|_H^2 \geq -c(x) |h(x)|_H + \eta + \frac{1}{2\alpha} |P(x, \alpha)|_H^2. \end{aligned}$$

Then $|P(x, \alpha)|_H^2 \leq 2\alpha(S(x) + c(x)|h(x)|_H - \eta)$ and $|P(x, \alpha)|_H \rightarrow 0$ as $\alpha \rightarrow 0^+$. Finally

$$S(x) = \lim_{\alpha \rightarrow 0^+} f_\alpha(x) = \lim_{\alpha \rightarrow 0^+} f(x + P(x, \alpha)) + \frac{1}{2\alpha} |P(x, \alpha)|_H^2 \geq \liminf_{\alpha \rightarrow 0^+} f(x + P(x, \alpha)) \geq f(x).$$

□

Proposition 3.3. *For $x \in X$ and $\alpha > 0$ let $P(x, \alpha)$ be the unique minimum point of the function $g_{\alpha, x}$, given by Proposition 3.1. For $p \in H$, we have $p = P(x, \alpha)$ if, and only if,*

$$(3.2) \quad f(x+p) \leq f(x+h) + \frac{1}{\alpha} \langle p, h-p \rangle_H,$$

for every $h \in H$.

Proof. Let $\beta \in (0, 1)$ and $h \in H$. Consider $p_\beta = \beta h + (1-\beta)P(x, \alpha)$ and observe that

$$\begin{aligned} f_\alpha(x) &= f(x + P(x, \alpha)) + \frac{1}{2\alpha} |P(x, \alpha)|_H^2 \leq f(x + p_\beta) + \frac{1}{2\alpha} |p_\beta|_H^2 \leq \\ &\leq \beta f(x+h) + (1-\beta)f(x + P(x, \alpha)) + \frac{\beta^2}{2\alpha} |h|_H^2 + \frac{\beta(1-\beta)}{\alpha} \langle P(x, \alpha), h \rangle_H + \frac{(1-\beta)^2}{2\alpha} |P(x, \alpha)|_H^2. \end{aligned}$$

Thus

$$\beta f(x + P(x, \alpha)) \leq \beta f(x+h) + \frac{\beta^2}{2\alpha} |h|_H^2 + \frac{\beta(1-\beta)}{\alpha} \langle P(x, \alpha), h \rangle_H + \frac{\beta(\beta-2)}{2\alpha} |P(x, \alpha)|_H^2.$$

Dividing by β we get

$$f(x + P(x, \alpha)) \leq f(x+h) + \frac{\beta}{2\alpha} |h|_H^2 + \frac{1-\beta}{\alpha} \langle P(x, \alpha), h \rangle_H + \frac{\beta-2}{2\alpha} |P(x, \alpha)|_H^2,$$

and letting $\beta \rightarrow 0^+$ we get

$$f(x + P(x, \alpha)) \leq f(x+h) + \frac{1}{\alpha} \langle P(x, \alpha), h - P(x, \alpha) \rangle_H.$$

Conversely, observe that if $p \in H$ satisfies inequality (3.2), then for every $h \in H$ we have

$$\begin{aligned} f(x+p) + \frac{1}{2\alpha} |p|_H^2 &\leq f(x+h) + \frac{1}{\alpha} \langle p, h-p \rangle_H + \frac{1}{2\alpha} |p|_H^2 \leq \\ &\leq f(x+h) + \frac{1}{\alpha} \langle p, h-p \rangle_H + \frac{1}{2\alpha} |p|_H^2 + \frac{1}{2\alpha} |h-p|_H^2 = f(x+h) + \frac{1}{2\alpha} |h|_H^2. \end{aligned}$$

□

Proposition 3.4. *Let $x \in X$ and $\alpha > 0$. The function $P_{x, \alpha} : H \rightarrow H$ defined as $P_{x, \alpha}(h) := P(x+h, \alpha)$ is Lipschitz continuous, with Lipschitz constant less or equal than 1.*

Proof. Let $\alpha > 0$, $x \in X$ and $h \in H$. By Proposition 3.3 we get

$$\begin{aligned} f(x + P(x, \alpha)) &\leq f(x+h + P(x+h, \alpha)) + \frac{1}{\alpha} \langle P(x, \alpha), h + P(x+h, \alpha) - P(x, \alpha) \rangle_H; \\ f(x+h + P(x+h, \alpha)) &\leq f(x + P(x, \alpha)) + \frac{1}{\alpha} \langle P(x+h, \alpha), P(x, \alpha) - h - P(x+h, \alpha) \rangle_H. \end{aligned}$$

Summing these inequalities and multiplying by α we get

$$\begin{aligned} 0 &\leq \langle P(x, \alpha), h + P(x+h, \alpha) - P(x, \alpha) \rangle_H + \langle P(x+h, \alpha), P(x, \alpha) - h - P(x+h, \alpha) \rangle_H = \\ &= -|P(x+h, \alpha) - P(x, \alpha)|_H^2 + \langle P(x, \alpha) - P(x+h, \alpha), h \rangle_H. \end{aligned}$$

Using the Cauchy–Schwartz inequality we get

$$|P(x+h, \alpha) - P(x, \alpha)|_H^2 \leq \langle P(x, \alpha) - P(x+h, \alpha), h \rangle_H \leq |P(x+h, \alpha) - P(x, \alpha)|_H |h|_H.$$

So $|P(x+h, \alpha) - P(x, \alpha)|_H \leq |h|_H$. \square

Proposition 3.5. *Let $\alpha > 0$. f_α is differentiable along H at every point $x \in X$. Moreover, for every $x \in X$, we have*

$$\nabla_H f_\alpha(x) = -\frac{1}{\alpha} P(x, \alpha).$$

Proof. By Proposition 3.3, for every $\alpha > 0$ and $h \in H$, we get

$$\begin{aligned} f_\alpha(x+h) - f_\alpha(x) &= \\ &= f(x+h+P(x+h, \alpha)) + \frac{1}{2\alpha} |P(x+h, \alpha)|_H^2 - f(x+P(x, \alpha)) - \frac{1}{2\alpha} |P(x, \alpha)|_H^2 \geq \\ &\geq \frac{1}{2\alpha} |P(x+h, \alpha)|_H^2 - \frac{1}{\alpha} \langle P(x, \alpha), h + P(x+h, \alpha) - P(x, \alpha) \rangle_H - \frac{1}{2\alpha} |P(x, \alpha)|_H^2 = \\ &= \frac{1}{2\alpha} |P(x+h, \alpha) - P(x, \alpha)|_H^2 - \frac{1}{\alpha} \langle P(x, \alpha), h \rangle_H \geq -\frac{1}{\alpha} \langle P(x, \alpha), h \rangle_H. \end{aligned}$$

In a similar way, for every $\alpha > 0$ and $h \in H$, we have

$$f_\alpha(x+h) - f_\alpha(x) \leq -\frac{1}{\alpha} \langle P(x+h, \alpha), h \rangle_H.$$

Combining these inequalities and applying Proposition 3.4 we get, for every $\alpha > 0$ and $h \in H$,

$$\begin{aligned} 0 \leq f_\alpha(x+h) - f_\alpha(x) + \frac{1}{\alpha} \langle P(x, \alpha), h \rangle_H &\leq \frac{1}{\alpha} \langle P(x, \alpha) - P(x+h, \alpha), h \rangle_H \leq \\ &\leq \frac{1}{\alpha} |P(x, \alpha) - P(x+h, \alpha)|_H |h|_H \leq \frac{1}{\alpha} |h|_H^2. \end{aligned}$$

So, for every $\alpha > 0$, f_α is differentiable along H at every point $x \in X$ and $\nabla_H f_\alpha(x) = -\frac{1}{\alpha} P(x, \alpha)$. \square

Proposition 3.6. *Let $\alpha > 0$ and $1 \leq p < +\infty$. If $f \in L^p(X, \mu)$, then $f_\alpha \in W^{2,p}(X, \mu)$.*

Proof. By Proposition 3.5 we get, for every $\alpha > 0$ and $x \in X$,

$$\nabla_H f_\alpha(x) = -\frac{1}{\alpha} P(x, \alpha).$$

Proposition 3.4 and [8, Theorem 5.11.2] give us that for every $\alpha > 0$, $\nabla_H f_\alpha \in W^{1,q}(X, \mu, H)$ for every $q \geq 1$. The conclusion follows from the inequality $f_\alpha(x) \leq f(x)$ for every $\alpha > 0$ and $x \in X$ (Proposition 3.2). \square

4. FINITE DIMENSIONAL RESULTS

In this section we recall some known finite dimensional results about the operator

$$(4.1) \quad L_\phi \psi = \sum_{i=1}^n D_{ii} \psi - \sum_{i=1}^n (D_i \phi + \xi_i) D_i \psi,$$

where ϕ is a convex function with Lipschitz continuous gradient, and $\psi \in \mathcal{C}_b^2(\mathbb{R}^n)$. We mainly refer to the results in [5]. We need a dimension-free uniform estimate for u and $\text{grad } u$, where u is a solution of

$$(4.2) \quad \lambda u(\xi) - L_\phi u(\xi) = f(\xi) \quad \xi \in \mathbb{R}^n,$$

where $\lambda > 0$ and f is a bounded γ -Hölder continuous function, for some $0 < \gamma < 1$. Recall that the space $\mathcal{C}_b^{k+\gamma}(\mathbb{R}^n)$, for $k \in \mathbb{N} \cup \{0\}$ and $0 < \gamma < 1$, is the space of the k -differentiable functions with bounded and γ -Hölder derivatives up to the order k , endowed with its standard norm (see [25, Section 2.7]), i.e. for $f \in \mathcal{C}_b^{k+\gamma}(\mathbb{R}^n)$ we let $\|f\|_{\mathcal{C}_b^{k+\gamma}(\mathbb{R}^n)} = \|f\|_{\mathcal{C}_b^k(\mathbb{R}^n)} + [D^k f]_\gamma$ where

$$[D^k f]_\gamma = \sum_{|\beta|=k} \sup \left\{ \frac{|D^\beta f(\xi_1) - D^\beta f(\xi_2)|}{|\xi_1 - \xi_2|^\gamma} \mid \xi_1, \xi_2 \in \mathbb{R}^n, \xi_1 \neq \xi_2 \right\}.$$

Also the space $\mathcal{C}^{k+\beta, m+\gamma}(A \times \mathbb{R}^n)$ for $k, m \in \mathbb{N} \cup \{0\}$, $0 < \beta, \gamma < 1$ and A an open subset of \mathbb{R} is the space of k -differentiable functions with β -Hölder derivatives up to the order k in the first variable and m -differentiable functions with γ -Hölder derivatives up to the order m in the second variable. As usual when we add the subscript *loc* we mean that the Hölder condition holds locally.

The following result will be useful.

Proposition 4.1. *Let $0 < \gamma < 1$ and assume that ϕ has a Lipschitz continuous gradient. For every $f \in \mathcal{C}_b^\gamma(\mathbb{R}^n)$ equation (4.2) has a unique solution $u \in \mathcal{C}_b^{2+\gamma}(\mathbb{R}^n)$, and there exists a constant $C > 0$, independent of f , such that*

$$(4.3) \quad \|u\|_{\mathcal{C}_b^{2+\gamma}(\mathbb{R}^n)} \leq C \|f\|_{\mathcal{C}_b^\gamma(\mathbb{R}^n)}.$$

Moreover if $\phi, f \in \mathcal{C}^\infty(\mathbb{R}^n)$, then $u \in \mathcal{C}^\infty(\mathbb{R}^n)$.

Inequality (4.3) was proved in [20, Theorem 1], and the local regularity result can be found in [17, Theorem 3.1.1].

Consider the problem

$$(4.4) \quad \begin{cases} D_t v(t, \xi) = L_\phi v(t, \xi) & t > 0, \xi \in \mathbb{R}^n; \\ v(0, \xi) = f(\xi), & \xi \in \mathbb{R}^n. \end{cases}$$

L_ϕ satisfies the conditions (2.1), (2.2), (2.3), and (2.4) of [5]. By [5, Theorem 3.1], for every $f \in \mathcal{C}_b(\mathbb{R}^n)$ there exists a unique bounded solution v of problem (4.4) belonging to $\mathcal{C}([0, \infty) \times \mathbb{R}^n) \cap \mathcal{C}_{\text{loc}}^{1+\gamma/2, 2+\gamma}((0, \infty) \times \mathbb{R}^n)$. If we set

$$(4.5) \quad T_t f(\xi) = v(t, \xi), \quad t \geq 0, \xi \in \mathbb{R}^n,$$

then $\{T_t\}_{t \geq 0}$ is a positive contraction semigroup on $\mathcal{C}_b(\mathbb{R}^n)$.

We want a dimension-free uniform estimate of the gradient of $T_t f$. Before proceeding we prove that the function $g(\xi) = |\xi|^2$ satisfies

$$(4.6) \quad \lim_{|\xi| \rightarrow +\infty} g(\xi) = +\infty \quad \text{and} \quad \sup_{\xi \in \mathbb{R}^n} L_\phi g(\xi) < +\infty.$$

A function g satisfying (4.6) is said to be a Lyapunov function for the operator L_ϕ . The first condition in (4.6) is obviously satisfied. Moreover

$$\begin{aligned} L_\phi g(\xi) &= 2n - 2\langle \text{grad } \phi(\xi), \xi \rangle - 2|\xi|^2 = 2n - 2\langle \text{grad } \phi(\xi) - \text{grad } \phi(0), \xi \rangle - 2\langle \text{grad } \phi(0), \xi \rangle - 2|\xi|^2 \leq \\ &\leq 2n + 2|\text{grad } \phi(0)||\xi| - 2|\xi|^2 = 2n + 2|\text{grad } \phi(0)||\xi| - 2|\xi|^2, \end{aligned}$$

where we have used the fact that $\langle \text{grad } \phi(\xi) - \text{grad } \phi(0), \xi \rangle \geq 0$ for every $\xi \in \mathbb{R}^n$ since ϕ is a differentiable convex function (see [23, Example 2.2(a)]). So the second condition in (4.6) is satisfied. This implies that g is a Lyapunov function and we get the following formulation of [19, Proposition 2.1].

Proposition 4.2. *Assume that $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ is convex and has Lipschitz continuous gradient. Let $T > 0$ and let $z_0 \in \mathcal{C}([0, T] \times \mathbb{R}^n)$ be a bounded function. Let $z \in \mathcal{C}([0, T] \times \mathbb{R}^n) \cap \mathcal{C}^{1,2}((0, T] \times \mathbb{R}^n)$ be a bounded function satisfying*

$$\begin{cases} D_t z(t, \xi) - L_\phi z(t, \xi) \leq 0, & 0 < t \leq T, \xi \in \mathbb{R}^n; \\ z(0, \xi) = z_0(\xi), & \xi \in \mathbb{R}^n. \end{cases}$$

If $\sup z > 0$, then

$$\sup_{\xi \in \mathbb{R}^n} z(t, \xi) \leq \sup_{\xi \in \mathbb{R}^n} z_0(\xi) \quad 0 \leq t \leq T.$$

The dimension-free uniform estimate of the gradient of $T_t f$ follows from an application of Bernstein's method, we give the proof just for the sake of completeness. More general results can be found in [5], [6], [7] and [19], where larger classes of operators are studied, but no explicit dimension-free uniform estimates of the gradient of $T_t f$ are emphasized.

Proposition 4.3. *Assume that $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ is convex and has Lipschitz continuous gradient. Then for every $t \geq 0$ and $\xi \in \mathbb{R}^n$ we have $|T_t f(\xi)| \leq \|f\|_\infty$ and*

$$|\text{grad } T_t f(\xi)| \leq \frac{\|f\|_\infty}{\sqrt{t}} \quad t > 0 \text{ and } \xi \in \mathbb{R}^n,$$

for every $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$.

Proof. If $f \equiv 0$ then the conclusion is obvious. So we can assume, without loss of generality, that $f \not\equiv 0$. We set

$$(4.7) \quad z(t, \xi) := |v(t, \xi)|^2 + t|\text{grad } v(t, \xi)|^2 \quad t > 0 \text{ and } \xi \in \mathbb{R}^n$$

where $v(t, \xi) = T_t f(\xi)$. From the general regularity theory of parabolic problems we get that v is smooth for $t \geq 0$. We claim that the function z satisfies the hypotheses of Proposition 4.2. Indeed

$$\begin{aligned} D_t z(t, \xi) &= 2v(t, \xi)D_t v(t, \xi) + |\text{grad } v(t, \xi)|^2 + 2t \sum_{i=1}^n D_i v(t, \xi) D_i D_t v(t, \xi) = \\ &= 2v(t, \xi)\Delta v(t, \xi) - 2v(t, \xi)\langle \text{grad } \phi(\xi) + \xi, \text{grad } v(t, \xi) \rangle + |\text{grad } v(t, \xi)|^2 + 2t\langle \text{grad}(\Delta v(t, \xi)), \text{grad } v(t, \xi) \rangle + \\ &\quad - 2t \sum_{i,j=1}^n (D_j(D_i \phi + \xi_i) D_i v(t, \xi) D_j v(t, \xi) + (D_i \phi + \xi_i) D_{ij} v(t, \xi) D_j v(t, \xi)). \end{aligned}$$

Now we compute $L_\phi z$. We have

$$\begin{aligned} L_\phi z(t, \xi) &= 2|\text{grad } v(t, \xi)|^2 + 2v(t, \xi)\Delta v(t, \xi) + 2t\langle \text{grad}(\Delta v(t, \xi)), \text{grad } v(t, \xi) \rangle + \\ &\quad + 2t \sum_{i,j=1}^n (D_{ij} v(t, \xi))^2 - 2v(t, \xi)\langle \text{grad } \phi(\xi) + \xi, \text{grad } v(t, \xi) \rangle - 2t \sum_{i,j=1}^n (D_i \phi + \xi_i) D_{ij} v(t, \xi) D_j v(t, \xi). \end{aligned}$$

Then we get

$$\begin{aligned} D_t z(t, \xi) - L_\phi z(t, \xi) &= -|\text{grad } v(t, \xi)|^2 - 2t \sum_{i,j=1}^n (D_{ij} v(t, \xi))^2 + \\ &\quad - 2t\langle D^2 \phi(\xi) \text{grad } v(t, \xi), \text{grad } v(t, \xi) \rangle - 2t|\text{grad } v(t, \xi)|^2. \end{aligned}$$

Since ϕ is a convex function, $D^2 \phi$ is positive-semidefinite matrix, and so

$$D_t z(t, \xi) - L_\phi z(t, \xi) \leq 0 \quad t > 0, \xi \in \mathbb{R}^n.$$

Let $T > 0$. Since $z(0, \xi) = (f(\xi))^2$, we can apply Proposition 4.2 and we get

$$\sup_{\xi \in \mathbb{R}^n} z(t, \xi) \leq \|f\|_\infty^2 \quad 0 \leq t \leq T.$$

By equation (4.5) and equation (4.7)

$$|\text{grad } T_t f(\xi)| \leq \frac{\|f\|_\infty}{\sqrt{t}} \quad 0 < t \leq T, \xi \in \mathbb{R}^n.$$

Since the above estimate does not depend on T we can conclude

$$|\text{grad } T_t f(\xi)| \leq \frac{\|f\|_\infty}{\sqrt{t}} \quad t > 0, \xi \in \mathbb{R}^n.$$

In the same way we get $|T_t f(\xi)| \leq \|f\|_\infty$ for every $t \geq 0$ and $\xi \in \mathbb{R}^n$. □

By [5, Proposition 3.2] and [24, Proposition 3.6] there exists an operator A whose resolvent is

$$(4.8) \quad R(\lambda, A)f(\xi) = \int_0^{+\infty} e^{-\lambda t} (T_t f)(\xi) dt \quad \xi \in \mathbb{R}^n.$$

By [5, Proposition 3.4] if $\psi \in \mathcal{C}_b^2(\mathbb{R}^n)$, then $A\psi = L_\phi \psi$.

Proposition 4.4. *Assume that $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ is convex and has Lipschitz continuous gradient. Let u be a classical solution of equation (4.2). Then*

$$|\text{grad } u(\xi)| \leq \sqrt{\frac{\pi}{\lambda}} \|f\|_\infty \quad \xi \in \mathbb{R}^n.$$

Furthermore $\|u\|_\infty \leq \lambda^{-1} \|f\|_\infty$.

Proof. The furthermore part follows from the contractivity of T_t and formula (4.8). By Proposition 4.3 we can differentiate under the integral sign in formula (4.8) and we get

$$\text{grad } u(\xi) = \int_0^{+\infty} e^{-\lambda t} (\text{grad } T_t f)(\xi) dt \quad \xi \in \mathbb{R}^n.$$

Moreover, for every $\xi \in \mathbb{R}^n$

$$|\text{grad } u(\xi)| \leq \int_0^{+\infty} \frac{e^{-\lambda t}}{\sqrt{t}} dt \|f\|_\infty = \sqrt{\frac{\pi}{\lambda}} \|f\|_\infty.$$

□

5. PASSING TO INFINITE DIMENSION

This section is devoted to prove Theorem 1.2. We start by showing that if $\nabla_H U$ is H -Lipschitz, then equation (1.1) has a unique strong solution, in the sense of Definition 1.3, and this solution satisfies the Sobolev regularity estimates listed in Theorem 1.2.

We need to recall some basic definitions that can be found in [8]. Let Y be a separable Banach space, we recall that a function $F : X \rightarrow Y$ is said to be H -Lipschitz if $C > 0$ exists such that

$$(5.1) \quad \|F(x+h) - F(x)\|_Y \leq C|h|_H,$$

for every $h \in H$ and μ -a.e. $x \in X$ (see [8, Section 4.5 and Section 5.11]). We denote by $P_n : X \rightarrow H$ the projection

$$P_n(x) = \sum_{i=1}^n \widehat{e}_i(x) e_i \quad \text{for every } x \in X,$$

where \widehat{e}_i belongs to X^* , for every $i \in \mathbb{N}$ (formula (2.1)). Let $\mu_n := \mu \circ P_n^{-1}$ and $\tilde{\mu}_n := \mu \circ (I - P_n)^{-1}$. Recall that both measures are non-degenerate, centered and Gaussian on $P_n X$ and $(I - P_n)X$ respectively, and

$$(5.2) \quad \tilde{H}_n = (I - P_n)(H),$$

is the Cameron–Martin space associated with the measure $\tilde{\mu}_n$ on $(I - P_n)X$. For the proofs of such results see [8, Theorem 3.7.3].

Let $f \in L^p(X, \mu)$ for some $p \geq 1$ and $n \in \mathbb{N}$. We denote by $\mathbb{E}_n f$ the conditional expectation of f , i.e. for every $x \in X$

$$\mathbb{E}_n f(x) = \int_X f(P_n x + (I - P_n)y) d\mu(y).$$

We recall in the following proposition the results in [8, Corollary 3.5.2 and Proposition 5.4.5]

Proposition 5.1. *Let $1 \leq p < +\infty$ and $f \in L^p(X, \mu)$. Then $\mathbb{E}_n f$ converges to f in $L^p(X, \mu)$ and μ -a.e. and for every $n \in \mathbb{N}$*

$$\|\mathbb{E}_n f\|_{L^p(X, \mu)} \leq \|f\|_{L^p(X, \mu)}.$$

Moreover if $f \in W^{1,p}(X, \mu)$, then $\mathbb{E}_n f$ converges to f in $W^{1,p}(X, \mu)$ and μ -a.e., for every $n \in \mathbb{N}$ we have $\|\mathbb{E}_n f\|_{W^{1,p}(X, \mu)} \leq \|f\|_{W^{1,p}(X, \mu)}$ and

$$\partial_i \mathbb{E}_n f = \begin{cases} \mathbb{E}_n \partial_i f & 1 \leq i \leq n; \\ 0 & i > n. \end{cases}$$

Finally the same results, with obvious modifications, are true if $f \in W^{2,p}(X, \mu)$.

5.1. The case where $\nabla_H U$ is H -Lipschitz. In this subsection we will assume the following hypothesis on the weight:

Hypothesis 5.2. Let $U : X \rightarrow \mathbb{R}$ be a function satisfying Hypothesis 1.1. Assume that U is differentiable along H at every point $x \in X$, and $\nabla_H U$ is H -Lipschitz. We will denote by $[\nabla_H U]_{H\text{-Lip}}$ the H -Lipschitz constant of $\nabla_H U$, i.e. the constant C in formula (5.1).

We recall that by [8, Theorem 5.11.2] we have $U \in W^{2,t}(X, \mu)$, where t is the same as in Hypothesis 1.1. Observe that every convex function in $\mathcal{FC}_b^2(X)$ and every continuous linear functional $x^* \in X^*$ satisfy Hypothesis 5.2.

Let $f \in \mathcal{FC}_b^\infty(X)$ be such that $f(x) = \varphi(\widehat{e}_1(x), \dots, \widehat{e}_{N_0}(x))$ for some $N_0 \in \mathbb{N}$ and $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^{N_0})$. Throughout the rest of this subsection we let $n > N_0$.

Proposition 5.3. Consider the function $\psi_n : \mathbb{R}^n \rightarrow \mathbb{R}$ defined as

$$\psi_n(\xi) := \int_X U \left(\sum_{i=1}^n \xi_i e_i + (I - P_n)y \right) d\mu(y),$$

where $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. Then ψ_n belongs to $\mathcal{C}^1(\mathbb{R}^n)$ and it has Lipschitz gradient with Lipschitz constant less or equal than the H -Lipschitz constant of $\nabla_H U$.

Proof. Let $\{d_i \mid i = 1, \dots, n\}$ be the canonical basis of \mathbb{R}^n . We will prove that ψ_n admits derivative along d_i for every $i = 1, \dots, n$ and that the gradient is Lipschitz continuous. This implies that ψ_n is continuous.

First of all we prove that for every $i \in \mathbb{N}$ the function $\mathbb{E}_n \partial_i U(x)$ is finite everywhere. For every $x \in X$

$$\begin{aligned} |\mathbb{E}_n \partial_i U(x)| &\leq \int_X |\partial_i U(P_n x + (I - P_n)y)| d\mu(y) \leq \\ &\leq \int_X |\partial_i U(P_n x + (I - P_n)y)|^2 d\mu(y) \leq \int_X |\nabla_H U(P_n x + (I - P_n)y)|_H^2 d\mu(y) \leq \\ &\leq 2 \int_X |\nabla_H U(P_n x + (I - P_n)y) - \nabla_H U(y - P_n y)|_H^2 d\mu(y) + 2 \int_X |\nabla_H U(y - P_n y)|_H^2 d\mu(y) \leq \\ &\leq 2[\nabla_H U]_{H\text{-Lip}}^2 \int_X |P_n x|_H^2 d\mu(y) + 2 \int_X |\nabla_H U(y - P_n y)|_H^2 d\mu(y) = \\ &= 2[\nabla_H U]_{H\text{-Lip}}^2 |P_n x|_H^2 + 2 \int_{(I - P_n)X} |\nabla_H U(z)|_H^2 d\tilde{\mu}_n(z). \end{aligned}$$

The last term of this chain of inequalities is finite, indeed $\nabla_H U$ is \tilde{H}_n -Lipschitz continuous, by formula (5.2), and the conclusion follows from [8, Theorem 5.11.2].

Since U is continuous and for every $x \in X$ the function U is differentiable along H at x , with H -Lipschitz gradient along H , then for every $x \in X$ and $h \in H$, the function $F_{x,h}(t) := U(x + th)$ belongs to $\mathcal{C}^1[0, 1]$. Indeed $F'_{x,h}(t) = \langle \nabla_H U(x + th), h \rangle_H$ and for every $t_1, t_2 \in (0, 1)$ we have $|F'_{x,h}(t_1) - F'_{x,h}(t_2)| \leq [\nabla_H U]_{H\text{-Lip}} |h|_H^2 |t_1 - t_2|$. So $F'_{x,h}$ is Lipschitz continuous. So by the fundamental theorem of calculus we get

$$(5.3) \quad U(x + h) - U(x) = \int_0^1 \langle \nabla_H U(x + th), h \rangle_H dt.$$

Now we get for every $i = 1, \dots, n$ and $s \in (0, 1)$

$$\begin{aligned}
& \left| \frac{\psi_n(\xi + sd_i) - \psi_n(\xi)}{s} - \mathbb{E}_n \partial_i U \left(\sum_{j=1}^n \xi_j e_j \right) \right| = \left| \frac{1}{s} \int_X U \left(\sum_{j=1}^n (\xi_j + s \delta_{ij}) e_j + (I - P_n)y \right) + \right. \\
& \quad \left. - U \left(\sum_{j=1}^n \xi_j e_j + (I - P_n)y \right) d\mu(y) - \mathbb{E}_n \partial_i U \left(\sum_{j=1}^n \xi_j e_j \right) \right| = \\
& = \left| \frac{1}{s} \int_X \int_0^1 \left\langle \nabla_H U \left(\sum_{j=1}^n \xi_j e_j + (I - P_n)y + st \cdot e_i \right), s \cdot e_i \right\rangle_H dt d\mu(y) - \mathbb{E}_n \partial_i U \left(\sum_{j=1}^n \xi_j e_j \right) \right| = \\
& = \left| \frac{1}{s} \int_X \int_0^1 \left\langle \nabla_H U \left(\sum_{j=1}^n \xi_j e_j + (I - P_n)y + st \cdot e_i \right) - \nabla_H U \left(\sum_{j=1}^n \xi_j e_j + (I - P_n)y \right), s \cdot e_i \right\rangle_H dt d\mu(y) \right| \leq \\
& \leq |s| [\nabla_H U]_{H\text{-Lip}} \int_X \int_0^1 t dt d\mu(y) = \frac{|s|}{2} [\nabla_H U]_{H\text{-Lip}},
\end{aligned}$$

which goes to zero as $s \rightarrow 0$. So $D_i \psi_n(\xi) = (\mathbb{E}_n \partial_i U)(\sum_{j=1}^n \xi_j e_j)$. Finally, for every $\xi, \eta \in \mathbb{R}^n$

$$\begin{aligned}
& |\text{grad } \psi_n(\xi) - \text{grad } \psi_n(\eta)|^2 = \sum_{i=1}^n |D_i \psi_n(\xi) - D_i \psi_n(\eta)|^2 = \\
& = \sum_{i=1}^n \left| \int_X \partial_i U \left(\sum_{j=1}^n \xi_j e_j + (I - P_n)y \right) - \partial_i U \left(\sum_{j=1}^n \eta_j e_j + (I - P_n)y \right) d\mu(y) \right|^2 \leq \\
& \leq \int_X \left| \nabla_H U \left(\sum_{j=1}^n \xi_j e_j + (I - P_n)y \right) - \nabla_H U \left(\sum_{j=1}^n \eta_j e_j + (I - P_n)y \right) \right|_H^2 d\mu(y) \leq \\
& \leq [\nabla_H U]_{H\text{-Lip}}^2 \int_X \sum_{i=1}^n (\xi_i - \eta_i)^2 d\mu(y) = [\nabla_H U]_{H\text{-Lip}}^2 |\xi - \eta|^2.
\end{aligned}$$

So we have $|\text{grad } \psi_n(\xi) - \text{grad } \psi_n(\eta)| \leq [\nabla_H U]_{H\text{-Lip}} |\xi - \eta|$, for every $\xi, \eta \in \mathbb{R}^n$. \square

Now we mollify the functions ψ_n . Fix $\varepsilon > 0$ and $\theta \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ with support contained in the unit ball and $\int_{\mathbb{R}^n} \theta(\xi) d\xi = 1$. Let

$$\psi_n^\varepsilon(\xi) = \int_{\mathbb{R}^n} \psi_n(\xi - \varepsilon \eta) \theta(\eta) d\eta.$$

Then ψ_n^ε is convex, it belongs to $\mathcal{C}_b^\infty(\mathbb{R}^n)$ and $\text{grad } \psi_n^\varepsilon$ is Lipschitz continuous. For $\lambda > 0$ consider the problem

$$(5.4) \quad \lambda v_n^\varepsilon(\xi) - \mathcal{L}_\nu^{(n, \varepsilon)} v_n^\varepsilon(\xi) = \varphi(\pi_{N_0} \xi), \quad \xi \in \mathbb{R}^n,$$

where $\pi_{N_0} : \mathbb{R}^n \rightarrow \mathbb{R}^{N_0}$ is the projection on the first N_0 coordinates, and $\mathcal{L}_\nu^{(n, \varepsilon)}$ is the following operator:

$$(5.5) \quad \mathcal{L}_\nu^{(n, \varepsilon)} v = \sum_{i=1}^n D_{ii} v - \sum_{i=1}^n (D_i \psi_n^\varepsilon + \xi_i) D_i v, \quad v \in \mathcal{C}_b^2(\mathbb{R}^n).$$

By Proposition 4.1 we know that equation (5.4) admits a unique solution v_n^ε belonging to $\mathcal{C}^\infty(\mathbb{R}^n) \cap \bigcup_{\gamma \in (0, 1)} \mathcal{C}_b^{2+\gamma}(\mathbb{R}^n)$.

Proposition 5.4. v_n^ε belongs to $\mathcal{C}_b^3(\mathbb{R}^n)$.

Proof. We just need to prove that the third order derivatives are bounded. We start by differentiating equation (5.4),

$$(1 + \lambda)D_j v_n^\varepsilon(\xi) - \mathcal{L}_\nu^{(n,\varepsilon)} D_j v_n^\varepsilon(\xi) = D_j \varphi(\pi_{N_0} \xi) - \sum_{i=1}^n D_j D_i \psi_n^\varepsilon(\xi) D_i v_n^\varepsilon(\xi) \quad \text{for } 1 \leq j \leq N_0;$$

$$(1 + \lambda)D_j v_n^\varepsilon(\xi) - \mathcal{L}_\nu^{(n,\varepsilon)} D_j v_n^\varepsilon(\xi) = - \sum_{i=1}^n D_j D_i \psi_n^\varepsilon(\xi) D_i v_n^\varepsilon(\xi) \quad \text{for } j = N_0 + 1, \dots, n.$$

In both equations the right hand side is Lipschitz continuous and bounded. By Proposition 4.1 we get $D_j v_n^\varepsilon \in \bigcup_{\gamma \in (0,1)} \mathcal{C}_b^{2+\gamma}(\mathbb{R}^n)$ for every $j = 1, \dots, n$. In particular v_n^ε belongs to $\mathcal{C}_b^3(\mathbb{R}^n)$. \square

We return to infinite dimension. Set

$$U_n^\varepsilon(x) := \psi_n^\varepsilon(\widehat{e}_1(x), \dots, \widehat{e}_n(x)), \quad V_n^\varepsilon(x) := v_n^\varepsilon(\widehat{e}_1(x), \dots, \widehat{e}_n(x)), \quad x \in X.$$

Let $\nu_n^\varepsilon = e^{-U_n^\varepsilon} \mu$. The operator $L_\nu^{(n,\varepsilon)}$ is defined as L , namely

$$D(L_\nu^{(n,\varepsilon)}) = \left\{ u \in W^{1,2}(X, \nu_n^\varepsilon) \mid \text{there exists } g \in L^2(X, \nu_n^\varepsilon) \text{ such that} \right. \\ \left. \int_X \langle \nabla_H u, \nabla_H \varrho \rangle_H d\nu_n^\varepsilon = - \int_X g \varrho d\nu_n^\varepsilon \text{ for every } \varrho \in \mathcal{F}\mathcal{C}_b^\infty(X) \right\},$$

and if $u \in D(L_\nu^{(n,\varepsilon)})$ we let $L_\nu^{(n,\varepsilon)} u = g$. It is easily seen that $\mathcal{F}\mathcal{C}_b^2(X) \subseteq D(L_\nu^{(n,\varepsilon)})$ for every $n \in \mathbb{N}$ and $\varepsilon > 0$. Furthermore if $Z \in \mathcal{F}\mathcal{C}_b^2(X)$ is such that $Z(x) = \omega(\widehat{e}_1(x), \dots, \widehat{e}_k(x))$ for some $k \in \mathbb{N}$ and $\omega \in \mathcal{C}_b^2(\mathbb{R}^k)$, then

$$(5.6) \quad L_\nu^{(n,\varepsilon)} Z = \sum_{i=1}^k \partial_{ii} Z - \sum_{i=1}^k (\partial_i U_n^\varepsilon + \widehat{e}_i) \partial_i Z.$$

Proposition 5.5. *Assume Hypothesis 5.2 holds. The function $V_n^\varepsilon \in D(L_\nu) \cap D(L_\nu^{(n,\varepsilon)})$. For every $x \in X$*

$$(5.7) \quad \lambda V_n^\varepsilon - L_\nu V_n^\varepsilon = f + \langle \nabla_H U - \nabla_H U_n^\varepsilon, \nabla_H V_n^\varepsilon \rangle_H;$$

$$(5.8) \quad L_\nu^{(n,\varepsilon)} V_n^\varepsilon(x) = \mathcal{L}_\nu^{(n,\varepsilon)} v_n^\varepsilon(\widehat{e}_1(x), \dots, \widehat{e}_n(x));$$

Moreover the following inequality holds for every $x \in X$

$$(5.9) \quad |\nabla_H V_n^\varepsilon(x)|_H \leq \sqrt{\frac{\pi}{\lambda}} \|\varphi\|_\infty.$$

Proof. Since $V_n^\varepsilon \in \mathcal{F}\mathcal{C}_b^3(X)$, $V_n^\varepsilon \in D(L_\nu) \cap D(L_\nu^{(n,\varepsilon)})$. Equality (5.7) and equality (5.8) follow from equality (5.5) and equality (5.6) and some computations. The moreover part is a consequence of Proposition 4.4. \square

Proposition 5.6. *Assume Hypothesis 5.2 holds. Then $\lambda V_n^{\frac{1}{n}} - L_\nu V_n^{\frac{1}{n}}$ converges to f in $L^2(X, \nu)$ as n goes to $+\infty$.*

Proof. Using equality (5.7) and inequality (5.9) we get

$$\begin{aligned} \int_X \left| \lambda V_n^{\frac{1}{n}}(x) - L_\nu V_n^{\frac{1}{n}}(x) - f(x) \right|^2 d\nu(x) &= \int_X \left| \left\langle \nabla_H U(x) - \nabla_H U_n^{\frac{1}{n}}(x), \nabla_H V_n^{\frac{1}{n}}(x) \right\rangle_H \right|^2 d\nu(x) \leq \\ &\leq \frac{\pi}{\lambda} \|\varphi\|_\infty^2 \int_X \left| \nabla_H U(x) - \nabla_H U_n^{\frac{1}{n}}(x) \right|_H^2 d\nu(x) \leq \\ &\leq \frac{\pi}{\lambda} \|\varphi\|_\infty^2 \left(\int_X |\nabla_H U(x) - \nabla_H \mathbb{E}_n U(x)|_H^2 d\nu(x) + \int_X \left| \nabla_H \mathbb{E}_n U(x) - \nabla_H U_n^{\frac{1}{n}}(x) \right|_H^2 d\nu(x) \right). \end{aligned}$$

We recall that due to Hypothesis 5.2 we have $t > 3$ and e^{-U} belongs to $L^{\frac{t}{t-2}}(X, \mu)$ (see the discussion after Hypothesis 1.1). Then

$$\int_X |\nabla_H U(x) - \nabla_H \mathbb{E}_n U(x)|_H^2 d\nu(x) \leq \|e^{-U}\|_{L^{\frac{t}{t-2}}(X, \mu)} \left(\int_X |\nabla_H U(x) - \nabla_H \mathbb{E}_n U(x)|_H^t d\mu(x) \right)^{\frac{2}{t}};$$

by Proposition 5.1 the integral in the right hand side vanishes as $n \rightarrow +\infty$.

Let $\mu_n = \mu \circ P_n^{-1}$ and let $[\text{grad } \psi_n]_1$ be the Lipschitz constant of $\text{grad } \psi_n$. By the change of variable formula (see [8, Formula (A.3.1)]) and Proposition 5.3 we get

$$\begin{aligned} & \int_X \left| \nabla_H \mathbb{E}_n U(x) - \nabla_H U_n^{\frac{1}{n}}(x) \right|_H^2 d\nu(x) \leq \\ & \leq \|e^{-U}\|_{L^{\frac{t}{t-2}}(X, \mu)} \left(\int_{\mathbb{R}^n} \left| \text{grad } \psi_n(\xi) - \text{grad } \psi_n^{\frac{1}{n}}(\xi) \right|^t d\mu_n(\xi) \right)^{\frac{2}{t}} \leq \\ & \leq \|e^{-U}\|_{L^{\frac{t}{t-2}}(X, \mu)} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\text{grad } \psi_n(\xi) - \text{grad } \psi_n(\xi - n^{-1}\eta)| \theta(\eta) d\eta \right)^t d\mu_n(\xi) \right)^{\frac{2}{t}} \leq \\ & \leq \|e^{-U}\|_{L^{\frac{t}{t-2}}(X, \mu)} \left(\frac{[\text{grad } \psi_n]_1}{n} \right)^2 \left(\int_{\mathbb{R}^n} |\eta| \theta(\eta) d\eta \right)^2 \leq \|e^{-U}\|_{L^{\frac{t}{t-2}}(X, \mu)} \left(\frac{[\nabla_H U]_{H\text{-Lip}}}{n} \right)^2. \end{aligned}$$

The last term of this chain of inequalities goes to zero as $n \rightarrow +\infty$ \square

Proposition 5.7. *Assume Hypothesis 5.2 holds. Then $(\lambda I - L_\nu)(\mathcal{FC}_b^3(X))$ is dense in $L^2(X, \nu)$.*

Proof. It follows from Proposition 5.6 and the density of the space $\mathcal{FC}_b^\infty(X)$ in $L^2(X, \nu)$ (see [15, Proposition 3.6]). \square

Proposition 5.8. *Assume Hypothesis 5.2 holds. For every $\lambda > 0$ and $f \in L^2(X, \nu)$, there exists a unique strong solution of equation (1.1) in the sense of Definition 1.3.*

Proof. First of all observe that $L_\nu : \mathcal{FC}_b^3(X) \rightarrow L^2(X, \nu)$ is a dissipative operator. Indeed, for every $u \in \mathcal{FC}_b^3(X)$, we have

$$\int_X u L_\nu u d\nu \leq 0.$$

Combining Proposition 5.7 and the Lumer–Phillips theorem (see [14, Theorem 2.3.15]), we get that the closure $\overline{L_\nu}$ of the operator L_ν generates a contraction semigroup and $\mathcal{FC}_b^3(X)$ is a core for $\overline{L_\nu}$, i.e. it is dense in $D(\overline{L_\nu})$ with the graph norm. In particular for every $\lambda > 0$ and $f \in L^2(X, \nu)$, equation (1.1) has a unique strong solution $u \in D(\overline{L_\nu})$. \square

We recall the following theorem (see [16, Theorem 3.1(2)]).

Theorem 5.9. *Let $F \in W^{2,p}(X, \mu)$, for some $p > 1$, be a convex function. Then $\nabla_H^2 F$ is a positive Hilbert–Schmidt operator μ -a.e., i.e. $\langle \nabla_H^2 F(x)h, h \rangle_H \geq 0$, for μ -a.e. $x \in X$ and every $h \in H$.*

We will state now a regularity result when U satisfies Hypothesis 5.2.

Theorem 5.10. *Let U be a function satisfying Hypothesis 5.2. Let $\lambda > 0$, $f \in L^2(X, \nu)$, and let u be the strong solution of equation (1.1). Then $u \in W^{2,2}(X, \nu)$ and*

$$(5.10) \quad \|u\|_{L^2(X, \nu)} \leq \frac{1}{\lambda} \|f\|_{L^2(X, \nu)}; \quad \|\nabla_H u\|_{L^2(X, \nu; H)} \leq \frac{1}{\sqrt{\lambda}} \|f\|_{L^2(X, \nu)};$$

$$(5.11) \quad \|\nabla_H^2 u\|_{L^2(X, \nu; \mathcal{H}_2)} \leq \sqrt{2} \|f\|_{L^2(X, \nu)}.$$

Moreover u is a weak solution of equation (1.1). Finally if $\{u_n\}_{n \in \mathbb{N}} \subseteq \mathcal{FC}_b^3(X)$ is a strong solution sequence for u (see Definition 1.3), then u_n converges to u in $W^{2,2}(X, \nu)$.

Proof. By Proposition 5.8 a sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq \mathcal{FC}_b^3(X)$ and a function $u \in D(\overline{L_\nu}) \subseteq W^{1,2}(X, \nu)$ exist such that u_n converges to u in $L^2(X, \nu)$ and

$$L^2(X, \nu)\text{-}\lim_{n \rightarrow +\infty} \lambda u_n - L_\nu u_n = f.$$

Let $f_n := \lambda u_n - L_\nu u_n$. By formula (2.5) and the fact that $U \in W^{2,t}(X, \mu)$, where t is the same as in Hypothesis 1.1, we get $f_n \in W^{1,2}(X, \nu)$. Multiplying by u_n and integrating we get

$$\begin{aligned} \int_X f_n(x) u_n(x) d\nu(x) &= \lambda \int_X u_n^2(x) d\nu(x) - \int_X u_n(x) L_\nu u_n(x) d\nu(x) = \\ &= \lambda \int_X u_n^2(x) d\nu(x) + \int_X |\nabla_H u_n(x)|_H^2 d\nu(x). \end{aligned}$$

Using the Cauchy-Schwarz inequality in the left hand side integral we get

$$(5.12) \quad \|u_n\|_{L^2(X, \nu)} \leq \frac{1}{\lambda} \|f_n\|_{L^2(X, \nu)}; \quad \|\nabla_H u_n\|_{L^2(X, \nu; H)} \leq \frac{1}{\sqrt{\lambda}} \|f_n\|_{L^2(X, \nu)}.$$

Since $\{u_n\}_{n \in \mathbb{N}}$ and $\{f_n\}_{n \in \mathbb{N}}$ converge to u and f , respectively, in $L^2(X, \nu)$ we get

$$\|u\|_{L^2(X, \nu)} = \lim_{n \rightarrow +\infty} \|u_n\|_{L^2(X, \nu)} \leq \lim_{n \rightarrow +\infty} \frac{1}{\lambda} \|f_n\|_{L^2(X, \nu)} = \frac{1}{\lambda} \|f\|_{L^2(X, \nu)}.$$

Moreover

$$\|\nabla_H u_n - \nabla_H u_m\|_{L^2(X, \nu; H)} \leq \frac{1}{\sqrt{\lambda}} \|f_n - f_m\|_{L^2(X, \nu)},$$

then $\{\nabla_H u_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(X, \nu; H)$. By the closability of ∇_H in $L^2(X, \nu)$ it follows that $u \in W^{1,2}(X, \nu)$ and

$$L^2(X, \nu; H)\text{-}\lim_{n \rightarrow +\infty} \nabla_H u_n = \nabla_H u.$$

Therefore

$$\|\nabla_H u\|_{L^2(X, \nu; H)} = \lim_{n \rightarrow +\infty} \|\nabla_H u_n\|_{L^2(X, \nu; H)} \leq \lim_{n \rightarrow +\infty} \frac{1}{\sqrt{\lambda}} \|f_n\|_{L^2(X, \nu)} = \frac{1}{\sqrt{\lambda}} \|f\|_{L^2(X, \nu)}.$$

Using formula (2.5), we differentiate the equality $\lambda u_n - L_\nu u_n = f_n$ with respect to the e_j direction, we multiply the result by $\partial_j u$, sum over j and finally integrate over X with respect to ν . We obtain

$$(1 + \lambda) \int_X |\nabla_H u_n|_H^2 d\nu + \int_X \|\nabla_H^2 u_n\|_{\mathcal{H}_2}^2 d\nu + \int_X \langle \nabla_H^2 U \nabla_H u_n, \nabla_H u_n \rangle_H d\nu = \int_X f_n^2 d\nu - \lambda \int_X f_n u_n d\nu.$$

Using inequalities (5.12) and Theorem 5.9 we get

$$\|\nabla_H^2 u_n\|_{L^2(X, \nu; \mathcal{H}_2)} \leq \sqrt{2} \|f_n\|_{L^2(X, \nu)}.$$

We remark that

$$\|\nabla_H^2 u_n - \nabla_H^2 u_m\|_{L^2(X, \nu; \mathcal{H}_2)} \leq \sqrt{2} \|f_n - f_m\|_{L^2(X, \nu)},$$

then $\{\nabla_H^2 u_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(X, \nu; \mathcal{H}_2)$. By the closability of ∇_H^2 in $L^2(X, \nu)$ it follows that $u \in W^{2,2}(X, \nu)$ and

$$L^2(X, \nu; \mathcal{H}_2)\text{-}\lim_{n \rightarrow +\infty} \nabla_H^2 u_n = \nabla_H^2 u.$$

Therefore

$$\|\nabla_H^2 u\|_{L^2(X, \nu; \mathcal{H}_2)} = \lim_{n \rightarrow +\infty} \|\nabla_H^2 u_n\|_{L^2(X, \nu; \mathcal{H}_2)} \leq \lim_{n \rightarrow +\infty} \sqrt{2} \|f_n\|_{L^2(X, \nu)} = \sqrt{2} \|f\|_{L^2(X, \nu)},$$

and $\{u_n\}_{n \in \mathbb{N}}$ converges to u in $W^{2,2}(X, \nu)$.

Now we want to show that u is a weak solution of equation (1.1). Let $\varphi \in \mathcal{FC}_b^\infty(X)$ and $n \in \mathbb{N}$, then

$$\lambda \int_X u_n \varphi d\nu - \int_X L_\nu u_n \varphi d\nu = \int_X f_n \varphi d\nu.$$

By the definition of L_ν , we get

$$(5.13) \quad \lambda \int_X u_n \varphi d\nu + \int_X \langle \nabla_H u_n, \nabla_H \varphi \rangle_H d\nu = \int_X f_n \varphi d\nu.$$

Since $\{u_n\}_{n \in \mathbb{N}}$ converges to u in $W^{2,2}(X, \nu)$ we obtain

$$\lim_{n \rightarrow +\infty} \lambda \int_X u_n \varphi d\nu = \lambda \int_X u \varphi d\nu,$$

and

$$\lim_{n \rightarrow +\infty} \int_X \langle \nabla_H u_n, \nabla_H \varphi \rangle_H d\nu = \int_X \langle \nabla_H u, \nabla_H \varphi \rangle_H d\nu.$$

Since $\{f_n\}_{n \in \mathbb{N}}$ converges to f in $L^2(X, \nu)$ we have

$$\lim_{n \rightarrow +\infty} \int_X f_n \varphi d\nu = \int_X f \varphi d\nu.$$

Then taking the limit as n goes to $+\infty$ in (5.13) we get that u is a weak solution of equation (1.1), i.e. for every $\varphi \in \mathcal{F}\mathcal{C}_b^\infty(X)$, we have $\lambda \int_X u \varphi d\nu + \int_X \langle \nabla_H u, \nabla_H \varphi \rangle_H d\nu = \int_X f \varphi d\nu$. \square

Remark 5.11. The hypothesis of continuity of the function U , in Hypothesis 5.2, can be replaced by the weaker hypothesis of H -continuity, i.e. for μ -a.e. $x \in X$

$$\lim_{H \ni h \rightarrow 0} U(x+h) = U(x).$$

Anyway we will use the results of this section for the Moreau–Yosida approximations along H of a function U satisfying Hypothesis 1.1, that are continuous in our case.

5.2. The general case. In this subsection we assume that U satisfies Hypothesis 1.1. In this case we do not know if there exists a strong solution of equation (1.1), but the Lax–Milgram theorem gives us a weak solution of equation (1.1).

Let $\alpha \in (0, 1]$ and let U_α be the Moreau–Yosida approximation along H of U , defined in Section 3. Consider the measure

$$(5.14) \quad \nu_\alpha = e^{-U_\alpha} \mu.$$

Proposition 5.12. *Let $\alpha \in (0, 1]$. U_α satisfies Hypothesis 5.2. Moreover $e^{-U_\alpha} \in W^{1,p}(X, \mu)$, for every $p \geq 1$, and $U_\alpha \in W^{2,t}(X, \mu)$, where t is given by Hypothesis 1.1.*

Proof. By Proposition 2.3 there exist $x^* \in X^*$ and $\eta \in \mathbb{R}$ such that $U_1(x) \geq x^*(x) + \eta$ for every $x \in X$. Then by Proposition 3.2, for every $x \in X$ we have

$$U_\alpha(x) \geq U_1(x) \geq x^*(x) + \eta.$$

So $e^{-U_\alpha(x)} \leq e^{-x^*(x) - \eta}$ for every $x \in X$. By the change of variable formula (see [8, Formula (A.3.1)]) we obtain

$$\int_X e^{-x^*(x) - \eta} d\mu(x) = e^{-\eta} \int_{\mathbb{R}} e^{-\xi} d\mu_{x^*}(\xi) < +\infty,$$

where $\mu_{x^*} = \mu \circ (x^*)^{-1}$. So $e^{-U_\alpha} \in L^p(X, \mu)$ for every $p \geq 1$. By the differentiability of U_α along H (see Proposition 3.5) we get $\nabla_H e^{-U_\alpha(x)} = -e^{-U_\alpha(x)} \nabla_H U_\alpha(x)$ for every $x \in X$. By Proposition 3.4 and [8, Theorem 5.11.2] we get $e^{-U_\alpha} \in W^{1,p}(X, \mu)$, for every $p \geq 1$. Finally $U_\alpha \in W^{2,t}(X, \mu)$, by Proposition 3.6.

Differentiability along H and the H -Lipschitzianity of $\nabla_H U_\alpha$ follow from Proposition 3.4 and Proposition 3.5. The convexity follows from the following standard argument: let $\varepsilon > 0$, $x_1, x_2 \in X$ and $\lambda \in [0, 1]$ and consider $h_\varepsilon(x_1), h_\varepsilon(x_2) \in H$ such that for $i = 1, 2$

$$U(x + h_\varepsilon(x_i)) + \frac{1}{2\alpha} |h_\varepsilon(x_i)|_H^2 \leq U_\alpha(x_i) + \varepsilon.$$

We get

$$\begin{aligned} & U_\alpha(\lambda x_1 + (1-\lambda)x_2) \leq \\ & \leq U(\lambda x_1 + (1-\lambda)x_2 + \lambda h_\varepsilon(x_1) + (1-\lambda)h_\varepsilon(x_2)) + \frac{1}{2\alpha} |\lambda h_\varepsilon(x_1) + (1-\lambda)h_\varepsilon(x_2)|_H^2 \leq \\ & \leq \lambda \left(U(x_1 + h_\varepsilon(x_1)) + \frac{1}{2\alpha} |h_\varepsilon(x_1)|_H^2 \right) + (1-\lambda) \left(U(x_2 + h_\varepsilon(x_2)) + \frac{1}{2\alpha} |h_\varepsilon(x_2)|_H^2 \right) \leq \\ & \leq \lambda U_\alpha(x_1) + (1-\lambda)U_\alpha(x_2) + \varepsilon. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ we get the convexity of U_α for every $\alpha \in (0, 1]$. Continuity of U_α is a consequence of Proposition 3.2 and [13, Corollary 2.4]. \square

Arguing as in [15, Proposition 4.2] and using Proposition 5.12 we get

$$\nabla_H : \mathcal{FC}_b^\infty(X) \rightarrow L^2(X, \nu_\alpha; H)$$

is a closable operator in $L^2(X, \nu_\alpha)$ for every $\alpha \in (0, 1]$. The same is true for the operator $(\nabla_H, \nabla_H^2) : \mathcal{FC}_b^\infty(X) \rightarrow L^2(X, \nu_\alpha; H) \times L^2(X, \nu_\alpha; \mathcal{H}_2)$ (see Proposition 2.1). In particular we can define the spaces $W^{1,2}(X, \nu_\alpha)$ and $W^{2,2}(X, \nu_\alpha)$ as the domains of their respective closures.

For $\alpha \in (0, 1]$, consider now the operator

$$D(L_{\nu_\alpha}) = \left\{ u \in W^{1,2}(X, \nu_\alpha) \mid \text{there exists } v \in L^2(X, \nu_\alpha) \text{ such that} \right. \\ \left. \int_X \langle \nabla_H u, \nabla_H \varphi \rangle d\nu_\alpha = - \int_X v \varphi d\nu_\alpha \text{ for every } \varphi \in \mathcal{FC}_b^\infty(X) \right\},$$

with $L_{\nu_\alpha} u = v$ if $u \in D(L_{\nu_\alpha})$.

Now we have all the tools needed to prove Theorem 1.2. The arguments are similar to those in [11, Theorem 3.9], we give the proof just for the sake of completeness.

Proof of Theorem 1.2. Let $\bar{f} \in \mathcal{FC}_b^\infty(X)$ and $\{\alpha_n\}_{n \in \mathbb{N}} \subseteq (0, 1]$ be a decreasing sequence converging to zero. Consider the family of equations

$$(5.15) \quad \lambda u_{\alpha_n} - L_{\nu_{\alpha_n}} u_{\alpha_n} = \bar{f}.$$

By Proposition 5.8 and Theorem 5.10, for every $n \in \mathbb{N}$, equation (5.15) has a unique strong solution, which coincides with the weak solution, $u_{\alpha_n} \in W^{2,2}(X, \nu_{\alpha_n})$ such that

$$(5.16) \quad \|u_{\alpha_n}\|_{L^2(X, \nu_{\alpha_n})} \leq \frac{1}{\lambda} \|\bar{f}\|_{L^2(X, \nu_{\alpha_n})}; \quad \|\nabla_H u_{\alpha_n}\|_{L^2(X, \nu_{\alpha_n}; H)} \leq \frac{1}{\sqrt{\lambda}} \|\bar{f}\|_{L^2(X, \nu_{\alpha_n})}; \\ \|\nabla_H^2 u_{\alpha_n}\|_{L^2(X, \nu_{\alpha_n}; \mathcal{H}_2)} \leq \sqrt{2} \|\bar{f}\|_{L^2(X, \nu_{\alpha_n})}.$$

By Proposition 3.2 and Proposition 5.12 we get, for every $n \in \mathbb{N}$,

$$(5.17) \quad \|\bar{f}\|_{L^2(X, \nu_{\alpha_n})}^2 \leq \int_X |\bar{f}(x)|^2 e^{-U_1(x)} d\mu(x) \leq \|\bar{f}\|_\infty^2 \int_X e^{-U_1(x)} d\mu(x) < +\infty.$$

By Proposition 3.2 we have $e^{-U} \leq e^{-U_{\alpha_n}}$. So the set $\{u_{\alpha_n} \mid n \in \mathbb{N}\}$ is bounded in $W^{2,2}(X, \nu)$.

By weak compactness a function $u \in W^{2,2}(X, \nu)$ and a subsequence, which we still denote by $\{u_{\alpha_n}\}_{n \in \mathbb{N}}$, exist such that $u_{\alpha_n} \rightarrow u$ weakly in $W^{2,2}(X, \nu)$ and $u_{\alpha_n}, \nabla_H u_{\alpha_n}, \nabla_H^2 u_{\alpha_n}$ converge pointwise μ -a.e. respectively to $u, \nabla_H u$ and $\nabla_H^2 u$.

By inequality (5.17) and the Lebesgue dominated convergence theorem we get

$$(5.18) \quad \lim_{n \rightarrow +\infty} \|\bar{f}\|_{L^2(X, \nu_{\alpha_n})} = \|\bar{f}\|_{L^2(X, \nu)}.$$

By the weak convergence of $\{u_{\alpha_n}\}_{n \in \mathbb{N}}$ in $W^{2,2}(X, \nu)$ to u , the lower semicontinuity of the norm of $L^2(X, \nu)$, $L^2(X, \nu; H)$ and $L^2(X, \nu; \mathcal{H}_2)$, inequalities (5.16) and equality (5.18) we have

$$\|u\|_{L^2(X, \nu)} \leq \liminf_{n \rightarrow \infty} \|u_{\alpha_n}\|_{L^2(X, \nu)} \leq \liminf_{n \rightarrow \infty} \|u_{\alpha_n}\|_{L^2(X, \nu_{\alpha_n})} \leq \frac{1}{\lambda} \liminf_{n \rightarrow \infty} \|\bar{f}\|_{L^2(X, \nu_{\alpha_n})} = \frac{1}{\lambda} \|\bar{f}\|_{L^2(X, \nu)}; \\ \|\nabla_H u\|_{L^2(X, \nu; H)} \leq \liminf_{n \rightarrow \infty} \|\nabla_H u_{\alpha_n}\|_{L^2(X, \nu; H)} \leq \liminf_{n \rightarrow \infty} \|\nabla_H u_{\alpha_n}\|_{L^2(X, \nu_{\alpha_n}; H)} \leq \\ \leq \frac{1}{\sqrt{\lambda}} \liminf_{n \rightarrow \infty} \|\bar{f}\|_{L^2(X, \nu_{\alpha_n})} = \frac{1}{\sqrt{\lambda}} \|\bar{f}\|_{L^2(X, \nu)};$$

and

$$\|\nabla_H^2 u\|_{L^2(X, \nu; \mathcal{H}_2)} \leq \liminf_{n \rightarrow \infty} \|\nabla_H^2 u_{\alpha_n}\|_{L^2(X, \nu; \mathcal{H}_2)} \leq \liminf_{n \rightarrow \infty} \|\nabla_H^2 u_{\alpha_n}\|_{L^2(X, \nu_{\alpha_n}; \mathcal{H}_2)} \leq \\ \leq \sqrt{2} \liminf_{n \rightarrow \infty} \|\bar{f}\|_{L^2(X, \nu_{\alpha_n})} = \sqrt{2} \|\bar{f}\|_{L^2(X, \nu)}.$$

Now we show that u is a weak solution of the equation

$$\lambda u - L_\nu u = \bar{f}.$$

We recall that $\{u_{\alpha_n}\}_{n \in \mathbb{N}}$ is a sequence of weak solutions of the equations (5.15), i.e.

$$(5.19) \quad \lambda \int_X u_{\alpha_n} \varphi d\nu_{\alpha_n} + \int_X \langle \nabla_H u_{\alpha_n}, \nabla_H \varphi \rangle_H d\nu_{\alpha_n} = \int_X \bar{f} \varphi d\nu_{\alpha_n}$$

for all $\varphi \in \mathcal{FC}_b^\infty(X)$ and $n \in \mathbb{N}$. By inequalities (5.16), for every $\varphi \in \mathcal{FC}_b^\infty(X)$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} \int_X |u_{\alpha_n} \varphi| e^{-U_{\alpha_n}} d\mu &\leq \|\varphi\|_\infty \int_X |u_{\alpha_n}| e^{-U_{\alpha_n}} d\mu \leq \|\varphi\|_\infty \left(\int_X u_{\alpha_n}^2 d\nu_{\alpha_n} \right)^{\frac{1}{2}} \left(\int_X e^{-U_{\alpha_n}} d\mu \right)^{\frac{1}{2}} \leq \\ &\leq \frac{1}{\lambda} \|\varphi\|_\infty \|\bar{f}\|_{L^2(X, \nu_{\alpha_n})} \left(\int_X e^{-U_1} d\mu \right)^{\frac{1}{2}} \leq \frac{1}{\lambda} \|\varphi\|_\infty \|\bar{f}\|_\infty \left(\int_X e^{-U_1} d\mu \right). \end{aligned}$$

Then by Proposition 3.2, Proposition 5.12, the pointwise μ -a.e. convergence of u_{α_n} to u , and the Lebesgue dominated convergence theorem we get

$$\lim_{n \rightarrow +\infty} \lambda \int_X u_{\alpha_n} \varphi d\nu_{\alpha_n} = \lambda \int_X u \varphi d\nu.$$

Similarly, for every $\varphi \in \mathcal{FC}_b^\infty(X)$ and $n \in \mathbb{N}$, by inequalities (5.16) we have

$$\begin{aligned} \int_X |\langle \nabla_H u_{\alpha_n}, \nabla_H \varphi \rangle_H| e^{-U_{\alpha_n}} d\mu &\leq \int_X |\nabla_H u_{\alpha_n}|_H |\nabla_H \varphi|_H e^{-U_{\alpha_n}} d\mu \leq \|\nabla_H \varphi\|_H \int_X |\nabla_H u_{\alpha_n}|_H e^{-U_{\alpha_n}} d\mu \leq \\ &\leq \|\nabla_H \varphi\|_H \left(\int_X |\nabla_H u_{\alpha_n}|_H^2 e^{-U_{\alpha_n}} d\mu \right)^{\frac{1}{2}} \left(\int_X e^{-U_{\alpha_n}} d\mu \right)^{\frac{1}{2}} \leq \\ &\leq \frac{1}{\sqrt{\lambda}} \|\nabla_H \varphi\|_H \|\bar{f}\|_{L^2(X, \nu_{\alpha_n})} \left(\int_X e^{-U_1} d\mu \right)^{\frac{1}{2}} \leq \frac{1}{\lambda} \|\nabla_H \varphi\|_H \|\bar{f}\|_\infty \int_X e^{-U_1} d\mu. \end{aligned}$$

Therefore by Proposition 3.2, Proposition 5.12, the pointwise μ -a.e. convergence of $\nabla_H u_{\alpha_n}$ to $\nabla_H u$, and the Lebesgue dominated convergence theorem we get

$$\lim_{n \rightarrow +\infty} \int_X \langle \nabla_H u_{\alpha_n}, \nabla_H \varphi \rangle_H d\nu_{\alpha_n} = \int_X \langle \nabla_H u, \nabla_H \varphi \rangle_H d\nu.$$

Finally, for every $\varphi \in \mathcal{FC}_b^\infty(X)$ and $n \in \mathbb{N}$, we get

$$\int_X \varphi \bar{f} e^{-U_{\alpha_n}} d\mu \leq \|\varphi\|_\infty \|\bar{f}\|_\infty \int_X e^{-U_1} d\mu.$$

Thus by Proposition 3.2, Proposition 5.12 and the Lebesgue dominated convergence theorem we get

$$\lim_{n \rightarrow +\infty} \int_X \varphi \bar{f} d\nu_{\alpha_n} = \int_X \varphi \bar{f} d\nu.$$

Taking the limit in equation (5.19) as $n \rightarrow +\infty$ we get the claim. If $f \in L^2(X, \nu)$, a standard density argument gives us the assertions of our theorem. \square

6. A CHARACTERIZATION OF THE DOMAIN OF L_ν : THE $\nabla_H U$ H -LIPSCHITZ CASE

We recall some basic facts about the divergence operator in weighted Gaussian spaces. For every measurable map $\Phi : X \rightarrow X$ and every $f \in \mathcal{FC}_b^\infty(X)$ we define

$$\partial_\Phi f(x) := \lim_{t \rightarrow 0} \frac{f(x + t\Phi(x)) - f(x)}{t},$$

whenever such limit exists. If the limit exists μ -a.e. in X and a function $g \in L^1(X, \nu)$ satisfies

$$(6.1) \quad \int_X \partial_\Phi f d\nu = - \int_X f g d\nu$$

for every $f \in \mathcal{FC}_b^\infty(X)$, then g is called weighted Gaussian divergence of Φ . Furthermore if g exists, then it is unique and it will be denoted by $\operatorname{div}_\nu \Phi := g$. Finally if $\Phi \in L^1(X, \nu; H)$ has weighted Gaussian divergence, then equality (6.1) becomes

$$\int_X \langle \nabla_H f, \Phi \rangle_H d\nu = - \int_X f \operatorname{div}_\nu \Phi d\nu$$

for every $f \in \mathcal{FC}_b^\infty(X)$. We will use the following result (see [15, Proposition 5.3]).

Proposition 6.1. *Let U be a function satisfying Hypothesis 1.1 such that $\nabla_H U$ is H -Lipschitz. Then every $\Phi \in W^{1,2}(X, \mu; H)$ has a weighted Gaussian divergence $\operatorname{div}_\nu \Phi \in L^2(X, \nu)$ and for every $f \in W^{1,2}(X, \nu)$ the following equality holds,*

$$\int_X \langle \nabla_H f, \Phi \rangle_H d\nu = - \int_X f \operatorname{div}_\nu \Phi d\nu.$$

Furthermore, if $\varphi_i = \langle \Phi, e_i \rangle_H$ for every $i \in \mathbb{N}$, then

$$\operatorname{div}_\nu \Phi = \sum_{i=1}^{+\infty} (\partial_i \varphi_i - \varphi_i \partial_i U - \varphi_i \widehat{e}_i),$$

where the series converges in $L^2(X, \nu)$. Finally $\|\operatorname{div}_\nu \Phi\|_{L^2(X, \nu)} \leq \|\Phi\|_{W^{1,2}(X, \nu; H)}$.

We are now able to prove a characterization result for the domain of L_ν .

Theorem 6.2. *Let U be a function satisfying Hypothesis 1.1 such that $\nabla_H U$ is H -Lipschitz. Then $D(L_\nu) = W^{2,2}(X, \nu)$. Moreover, for every $u \in D(L_\nu)$, it holds $L_\nu u = \operatorname{div}_\nu \nabla_H u$ and*

$$(6.2) \quad \|u\|_{D(L_\nu)} \leq \|u\|_{W^{2,2}(X, \nu)} \leq (2 + \sqrt{2}) \|u\|_{D(L_\nu)},$$

where $\|\cdot\|_{D(L_\nu)}$ is defined in formula (1.4).

Proof. Let $u \in D(L_\nu)$. We have $u - L_\nu u \in L^2(X, \nu)$. Then by Theorem 1.2 we get that $u \in W^{2,2}(X, \nu)$. Let $u \in W^{2,2}(X, \nu)$, by Proposition 6.1 we get that $\operatorname{div}_\nu \nabla_H u \in L^2(X, \nu)$ and

$$\int_X \langle \nabla_H f, \nabla_H u \rangle_H d\nu = - \int_X f \operatorname{div}_\nu \nabla_H u d\nu$$

for every $f \in \mathcal{FC}_b^\infty(X)$. Then we have $u \in D(L_\nu)$ and $L_\nu u = \operatorname{div}_\nu \nabla_H u$.

By Proposition 6.1 we have

$$(6.3) \quad \|u\|_{D(L_\nu)} = \|u\|_{L^2(X, \nu)} + \|L_\nu u\|_{L^2(X, \nu)} = \|u\|_{L^2(X, \nu)} + \|\operatorname{div}_\nu \nabla_H u\|_{L^2(X, \nu)} \leq \|u\|_{W^{2,2}(X, \nu)},$$

for every $u \in D(L_\nu)$. Now if $u \in D(L_\nu)$, then for every $\lambda \in (0, 1)$ the function $\lambda u - L_\nu u$ belongs to $L^2(X, \nu)$ and by Theorem 1.2 we get for $u \in D(L_\nu)$

$$(6.4) \quad \begin{aligned} \|u\|_{W^{2,2}(X, \nu)} &\leq \left(\frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} + \sqrt{2} \right) \|\lambda u - L_\nu u\|_{L^2(X, \nu)} \leq \left(\frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} + \sqrt{2} \right) \left(\lambda \|u\|_{L^2(X, \nu)} + \|L_\nu u\|_{L^2(X, \nu)} \right) \leq \\ &\leq \left(\frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} + \sqrt{2} \right) \left(\|u\|_{L^2(X, \nu)} + \|L_\nu u\|_{L^2(X, \nu)} \right) = \left(\frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}} + \sqrt{2} \right) \|u\|_{D(L_\nu)}. \end{aligned}$$

Letting $\lambda \rightarrow 1^-$ in inequality (6.4) we get

$$(6.5) \quad \|u\|_{W^{2,2}(X, \nu)} \leq (2 + \sqrt{2}) \|u\|_{D(L_\nu)}.$$

Combining inequality (6.3) and inequality (6.5), we get inequality (6.2). \square

Remark 6.3. By the proof of Theorem 6.2, if U satisfies Hypothesis 1.1 then

$$(6.6) \quad D(L_\nu) \subseteq W^{2,2}(X, \nu),$$

and inequality (6.5) holds for every $u \in D(L_\nu)$. We do not know if the additional assumption that $\nabla_H U$ is H -Lipschitz is necessary to guarantee the equality in formula (6.6) and inequality (6.3).

7. EXAMPLES

In this section we will denote by $d\xi$ the Lebesgue measure on $[0, 1]$. We recall that a function $f : X \rightarrow \mathbb{R}$ from a Banach space X to \mathbb{R} is Gâteaux differentiable at $x \in X$ if for every $y \in X$ the limit

$$(7.1) \quad \lim_{t \rightarrow 0} \frac{f(x + ty) - f(x)}{t}$$

exists and defines a linear (in y) map $f'(x)(\cdot)$ which is continuous from X to \mathbb{R} . Furthermore if the limit (7.1) exists uniformly for $y \in X$ such that $\|y\|_X = 1$, then the function f is said to be Fréchet differentiable. We will use the following result of Aronszajn (see [3, Theorem 1 of Chapter 2] and [22, Theorem 6]).

Theorem 7.1. *Suppose that X is a separable real Banach space. If $f : X \rightarrow \mathbb{R}$ is a continuous convex function, then f is Gâteaux differentiable outside of a Gaussian null set, i.e. a Borel set $A \subseteq X$ such that $\mu(A) = 0$ for every non-degenerate Gaussian measure μ on X .*

Consider the classical Wiener measure P^W on $\mathcal{C}[0, 1]$ (see [8, Example 2.3.11 and Remark 2.3.13] for its construction). Recall that the Cameron–Martin space H is the space of the continuous functions f on $[0, 1]$ such that f is absolutely continuous, $f' \in L^2([0, 1], d\xi)$ and $f(0) = 0$. In addition if $f, g \in H$, then $|f|_H = \|f'\|_{L^2([0, 1], d\xi)}$ and

$$\langle f, g \rangle_H = \int_0^1 f'(\xi)g'(\xi)d\xi,$$

see [8, Lemma 2.3.14]. An orthonormal basis of $L^2([0, 1], d\xi)$ is given by the functions

$$e_n(\xi) = \sqrt{2} \sin \frac{\xi}{\sqrt{\lambda_n}} \quad \text{where } \lambda_n = \frac{4}{\pi^2(2n-1)^2} \text{ for every } n \in \mathbb{N}.$$

We recall that P^W is a centered Gaussian measure on $\mathcal{C}_0[0, 1] = \{f \in \mathcal{C}[0, 1] \mid f(0) = 0\}$, and if $f \in H$, then

$$(7.2) \quad |f|_H^2 = \sum_{i=1}^{+\infty} \lambda_i^{-1} \langle f, e_i \rangle_{L^2([0, 1], d\xi)}^2.$$

7.1. A weight bounded from below. Let $U(f) = \int_0^1 f^2(\xi)d\xi$. Such function is convex, continuous and Fréchet differentiable at any $f \in \mathcal{C}_0[0, 1]$. Moreover for every $g \in \mathcal{C}_0[0, 1]$

$$(7.3) \quad U'(f)(g) = 2 \int_0^1 f(\xi)g(\xi)d\xi.$$

By the fact that $U(f) \leq \|f\|_\infty^2$ and the Fernique theorem [8, Theorem 2.8.5], we get that $U \in L^t(\mathcal{C}_0[0, 1], P^W)$ for every $t \geq 1$. By formula (7.3) we get

$$\nabla_H U(f) = 2 \sum_{i=1}^{+\infty} \left(\int_0^1 f(\xi)e_i(\xi)d\xi \right) e_i.$$

By [8, Proposition 5.4.6] if we show that $\nabla_H U$ is integrable for every $t \geq 1$ we get that $U \in W^{1,t}(\mathcal{C}_0[0, 1], P^W)$. We claim that $\nabla_H U$ is H -Lipschitz. Indeed, for $f \in \mathcal{C}_0[0, 1]$ and $h \in H$, we have

$$\begin{aligned} |\nabla_H U(f+h) - \nabla_H U(f)|_H^2 &= 4 \sum_{i=1}^{+\infty} |\partial_i U(f+h) - \partial_i U(f)|^2 = \\ &= 4 \sum_{i=1}^{+\infty} \left| \int_0^1 (f(\xi) + h(\xi))e_i(\xi)d\xi - \int_0^1 h(\xi)e_i(\xi)d\xi \right|^2 = 4 \sum_{i=1}^{+\infty} \left| \int_0^1 f(\xi)e_i(\xi)d\xi \right|^2 = 4 \sum_{i=1}^{+\infty} \langle f, e_i \rangle_{L^2([0, 1], d\xi)}^2. \end{aligned}$$

Observing that $\lambda_i \leq 4\pi^{-2}$ for every $i \in \mathbb{N}$, we get

$$\begin{aligned} |\nabla_H U(f+h) - \nabla_H U(f)|_H^2 &= 4 \sum_{i=1}^{+\infty} \langle h, e_i \rangle_{L^2([0,1], d\xi)}^2 = 4 \sum_{i=1}^{+\infty} \frac{\lambda_i}{\lambda_i} \langle h, e_i \rangle_{L^2([0,1], d\xi)}^2 \leq \\ &\leq \frac{16}{\pi^2} \sum_{i=1}^{+\infty} \lambda_i^{-1} \langle h, e_i \rangle_{L^2([0,1], d\xi)}^2 = \frac{16}{\pi^2} |h|_H^2 \end{aligned}$$

where the last equality follows from formula (7.2). According to [8, Theorem 5.11.2] $\nabla_H U$ belongs to $W^{1,t}(\mathcal{C}_0[0,1], P^W; H)$ for every $t > 1$. Thus U satisfies Hypothesis 5.2.

Theorem 5.10 can be applied in this case. So for every $f \in L^2(\mathcal{C}_0[0,1], e^{-U} P^W)$ and $\lambda > 0$ there exists a unique weak solution (which is also a strong solution, in the sense of Definition 1.3) $u \in W^{2,2}(\mathcal{C}_0[0,1], e^{-U} P^W)$ of equation (1.1) and u satisfies the Sobolev regularity estimates of Theorem 1.2. Furthermore we can apply Theorem 6.2 and get

$$D(L_{e^{-U} P^W}) = W^{2,2}(\mathcal{C}_0[0,1], e^{-U} P^W).$$

Finally, for every $v \in D(L_{e^{-U} P^W})$, we have $L_{e^{-U} P^W} v = \operatorname{div}_{e^{-U} P^W} \nabla_H u$ and

$$\|v\|_{D(L_{e^{-U} P^W})} \leq \|v\|_{W^{2,2}(\mathcal{C}_0[0,1], e^{-U} P^W)} \leq (2 + \sqrt{2}) \|v\|_{D(L_{e^{-U} P^W})},$$

where $\|\cdot\|_{D(L_{e^{-U} P^W})}$ is defined in formula (1.4).

7.2. A unbounded weight. Let $F(f) = \max_{\xi \in [0,1]} f(\xi)$ for $f \in \mathcal{C}[0,1]$. In order to compute the Gâteaux derivative of F we will use some classical arguments.

Proposition 7.2. *F is Lipschitz continuous, convex and Gâteaux differentiable at $f \in \mathcal{C}_0[0,1]$ if, and only if, $f \in M$ where*

$$M = \{f \in \mathcal{C}_0[0,1] \mid \text{there exists a unique } \xi_f \in [0,1] \text{ such that } F(f) = f(\xi_f)\}.$$

Furthermore, if $f \in M$ and ξ_f is the unique maximum point of f , then $F'(f)(g) = g(\xi_f)$.

Proof. Convexity and Lipschitz continuity are obvious. Let ξ_f be the unique maximum point of $f \in M$ and for $t \in \mathbb{R}$ and $g \in \mathcal{C}_0[0,1]$ choose $\xi_t \in [0,1]$ such that $F(f+tg) = f(\xi_t) + tg(\xi_t)$. Observe that

$$0 \leq f(\xi_f) - f(\xi_t) = f(\xi_f) + tg(\xi_f) - tg(\xi_f) - f(\xi_t) \leq f(\xi_t) + tg(\xi_t) - tg(\xi_f) - f(\xi_t) = t(g(\xi_t) - g(\xi_f)).$$

So we have

$$|f(\xi_f) - f(\xi_t)| \leq t(g(\xi_t) - g(\xi_f)) \leq 2|t| \|g\|_\infty.$$

Since $f \in M$ we have that $\xi_t \rightarrow \xi_f$ for $t \rightarrow 0$. Observe that

$$(7.4) \quad F(f+tg) - F(f) - tg(\xi_f) \geq f(\xi_f) + tg(\xi_f) - f(\xi_f) - tg(\xi_f) = 0;$$

and

$$(7.5) \quad F(f+tg) - F(f) - tg(\xi_f) \leq f(\xi_t) + tg(\xi_t) - f(\xi_t) - tg(\xi_f) \leq t(g(\xi_t) - g(\xi_f)) \leq |t| |g(\xi_t) - g(\xi_f)|.$$

By inequality (7.4) and inequality (7.5) we have that if $f \in M$, then F is Gâteaux differentiable at f and $F'(f)(g) = g(\xi_f)$.

Assume now that F is Gâteaux differentiable at $f \in \mathcal{C}_0[0,1] \setminus M$. Let $\xi_1, \xi_2 \in [0,1]$ such that $F(f) = f(\xi_1) = f(\xi_2)$ and $\xi_1 \neq \xi_2$. Set $g_1(\xi) = |\xi - \xi_1|$ and observe that

$$F(f+tg_1) - F(f) \geq f(\xi_2) + tg_1(\xi_2) - f(\xi_2) = t|\xi_2 - \xi_1|;$$

$$F(f+tg_1) - F(f) \geq f(\xi_1) + tg_1(\xi_1) - f(\xi_1) = 0.$$

These inequalities gives us the following contradiction:

$$\limsup_{t \rightarrow 0^-} \frac{F(f+tg_1) - F(f)}{t} \leq 0, \quad \liminf_{t \rightarrow 0^+} \frac{F(f+tg_1) - F(f)}{t} \geq |\xi_2 - \xi_1| > 0.$$

□

Let δ_1 be the Dirac measure concentrated in 1 and consider the weight e^{-U} where, for $f \in \mathcal{C}_0[0, 1]$,

$$U(f) = F(f) + \delta_1(f).$$

By Proposition 7.2, U is Lipschitz continuous and convex, and it is easy to show that U is unbounded, both from above and from below.

According to [8, Theorem 5.11.2] $U \in W^{1,t}(\mathcal{C}_0[0, 1], P^W)$ for every $t > 1$. So U satisfies Hypothesis 1.1. Furthermore by [8, Definition 5.2.3 and Proposition 5.4.6(iii)], Theorem 7.1 and by Proposition [15, Proposition 4.6] it can be seen that

$$\nabla_H U(f) = \sum_{i=1}^{+\infty} (e_i(\xi_f) + e_i(1))e_i \quad \text{for } P^W\text{-a.e. } f \in \mathcal{C}_0[0, 1].$$

Theorem 1.2 can be applied in this case. So for every $f \in L^2(\mathcal{C}_0[0, 1], e^{-U} P^W)$ and $\lambda > 0$ there exists a unique weak solution $u \in W^{2,2}(\mathcal{C}_0[0, 1], e^{-U} P^W)$ of equation (1.1). In addition

$$\begin{aligned} \|u\|_{L^2(\mathcal{C}_0[0,1], e^{-U} P^W)} &\leq \frac{1}{\lambda} \|f\|_{L^2(\mathcal{C}_0[0,1], e^{-U} P^W)}; \quad \|\nabla_H u\|_{L^2(\mathcal{C}_0[0,1], e^{-U} P^W; H)} \leq \frac{1}{\sqrt{\lambda}} \|f\|_{L^2(\mathcal{C}_0[0,1], e^{-U} P^W)}; \\ \|\nabla_H^2 u\|_{L^2(\mathcal{C}_0[0,1], e^{-U} P^W; \mathcal{H}_2)} &\leq \sqrt{2} \|f\|_{L^2(\mathcal{C}_0[0,1], e^{-U} P^W)}. \end{aligned}$$

Moreover by Remark 6.3, we obtain that $D(L_{e^{-U} P^W}) \subseteq W^{2,2}(\mathcal{C}_0[0, 1], e^{-U} P^W)$, and for every $v \in D(L_{e^{-U} P^W})$ we have

$$\|v\|_{W^{2,2}(\mathcal{C}_0[0,1], e^{-U} P^W)} \leq (2 + \sqrt{2}) \|v\|_{D(L_{e^{-U} P^W})},$$

where $\|\cdot\|_{D(L_{e^{-U} P^W})}$ is defined in formula (1.4).

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